Chapter 6 - Random Processes

Recall that a random variable X is a mapping between the sample space $\boldsymbol{S}$ and the extended real line $\mathcal{R}^+$. That is, $X : \boldsymbol{S} \rightarrow \mathcal{R}^+$.

A ***random process*** (a.k.a ***stochastic process***) is a mapping from the sample space into an ensemble of time functions (known as ***sample functions***). To every $\rho \in \boldsymbol{S}$, there corresponds a function of time (a ***sample function***) $X(t;\rho)$. This is illustrated by Figure 6-1. Often, from the notation, we drop the ρ variable, and write just X(t). However, the sample space ρ variable is always there, even if it is not shown explicitly.

For a fixed $t = t_0$, the quantity $X(t_0;\rho)$ is a random variable mapping sample space $\boldsymbol{S}$ into the real line. For fixed $\rho_0 \in \boldsymbol{S}$, the quantity $X(t;\rho_0)$ is a well-defined, non-random, function of time. Finally, for fixed t_0 and ρ_0, the quantity $X(t_0;\rho_0)$ is a real number.

Example 6-1: X maps Heads and Tails

Consider the coin tossing experiment where $\boldsymbol{S}$ = {H, T}. Define the random function

X(t;Heads) = sin(t)

X(t;Tails) = cos(t)

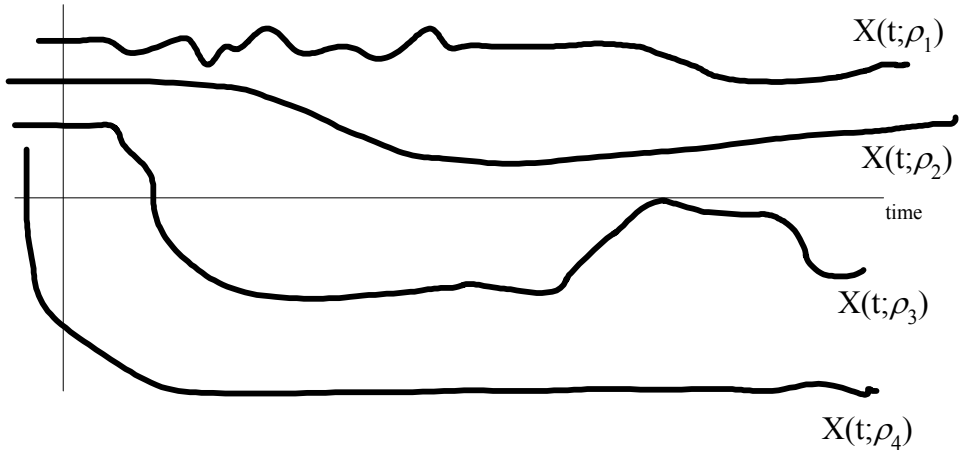

Figure 6-1: Sample functions of a random process.

Continuous and Discrete Random Processes

For a ***continuous random process***, probabilistic variable ρ takes on a continuum of values. For every fixed value $t = t_0$ of time, $X(t_0;\rho)$ is a continuous random variable.

Example 6-2: Let random variable A be uniform in [0, 1]. Define the continuous random process $X(t;\rho) = A(\rho)s(t)$, where s(t) is a unit-amplitude, T-periodic square wave. Notice that sample functions contain periodically-spaced (in time) jump discontinuities. However, the process is continuous.

For a ***discrete random process***, probabilistic variable ρ takes on only discrete values. For every fixed value $t = t_0$ of time, $X(t_0;\rho)$ is a discrete random variable.

Example 6-3: Consider the coin tossing experiment with $\boldsymbol{S}$ = {H, T}. Then X(t;H) = sin(t), X(t;T) = cos(t) defines a discrete random process. Notice that the sample functions are continuous functions of time. However, the process is discrete.

Distribution and Density Functions

The ***first-order distribution function*** is defined as

$$F(x,t) = \mathbf{P}[X(t) \le x]. \tag{6-1}$$

The ***first-order density function*** is defined as

$$f(x;t) \equiv \frac{dF(x,t)}{dx}. \tag{6-2}$$

These definitions generalize to the n^{th}-order case. For any given positive integer n, let x_1, x_2, ... , x_n denote n "realization" variables, and let t_1, t_2, ... , t_n denote n time variables. Then, define the ***n^{th}-order distribution function*** as

$$F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \mathbf{P}[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n]. \tag{6-3}$$

Similarly, define the ***n^th^-order density function*** as

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)}{\partial x_1 \partial x_2 \ldots \partial x_n} \tag{6-4}$$

In general, ***a complete statistical description of a random process requires knowledge of all order distribution functions.***

Stationary Random Process

A process X(t) is said to be ***stationary*** if its statistical properties do not change with time. More precisely, process X(t) is stationary if

$$F(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F(x_1, x_2, \ldots, x_n; t_1+c, t_2+c, \ldots, t_n+c) \tag{6-5}$$

for all orders n and all time shifts c.

Stationarity influences the form of the first- and second-order distribution/density functions. Let X(t) be stationary, so that

$$F(x; t) = F(x; t+c) \tag{6-6}$$

for all c. This implies that the first-order distribution function is independent of time. A similar statement can be made concerning the first-order density function. Now, consider the second-order distribution $F(x_1,x_2;t_1,t_2)$ of stationary X(t); for all t_1, t_2 and c, this function has the property

$$\begin{aligned} F(x_1, x_2; t_1, t_2) &= F(x_1, x_2; t_1+c, t_2+c) \\ &= F(x_1, x_2; 0, +\tau) \text{ if } c = -t_1 \\ &= F(x_1, x_2; -\tau, 0) \text{ if } c = -t_2 \end{aligned} \quad \Big\} \text{ where } \tau \equiv t_2 - t_1 . \tag{6-7}$$

Equation (6-7) must be true for all t_1, t_2 and c. Hence, the second-order distribution ***cannot depend on absolute*** t_1 and t_2; instead, $F(x_1,x_2;t_1,t_2)$ depends on the ***time difference*** $\tau \equiv t_2 - t_1$. In $F(x_1,x_2;t_1,t_2)$, you will only see t_1 and t_2 appear together as $t_2 - t_1$, which we define as τ. Often, for stationary processes, we change the notation and define

$$\underbrace{F(x_1, x_2; \tau)}_{\text{"new" notation}} \equiv \underbrace{F(x_1, x_2; t_1, t_1+\tau)}_{\text{"old" notation}}. \tag{6-8}$$

Similar statements can be made concerning the second-order density function.

Be careful! These conditions on first-order F(x) and second-order $F(x_1, x_2;\tau)$ are necessary conditions; ***they are not sufficient to imply stationarity***. For a given random process, suppose that the first order distribution/density is independent of time and the second-order distribution/density depends only on the time difference. Based on this knowledge alone, we cannot conclude that X(t) is stationary.

First- and Second-Order Probabilistic Averages

First- and second-order statistical averages are useful. The ***expected value*** of general random process X(t) is defined as

$$\eta(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f(x;t)\, dx. \tag{6-9}$$

In general, this is a time-varying quantity. The expected value is often called a ***first-order statistic*** since it depends on a first-order density function. The ***autocorrelation*** function of X(t) is defined as

$$R(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;t_1,t_2)\, dx_1\, dx_2. \tag{6-10}$$

In general, R depends on two time variables, t_1 and t_2. Also, R is an example of a ***second-order***

statistic since it depends on a second-order density function.

Suppose X(t) is stationary. Then the mean

$$\eta = E[X(t)] = \int_{-\infty}^{\infty} x\, f(x)\, dx \tag{6-11}$$

is constant, and the autocorrelation function

$$R(\tau) = E[X(t)X(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1, x_2; \tau)\, dx_1\, dx_2 \tag{6-12}$$

depends only on the time difference $\tau = t_2 - t_1$ (it does not depend on absolute time). However, the converse is not true: the conditions η a constant and $R(\tau)$ independent of absolute time do not imply that X(t) is stationary.

Wide Sense Stationarity (WSS)

Process X(t) is said to be ***wide-sense stationary*** (WSS) if

1) Mean $\eta = E[X(t)]$ is constant, and

2) Autocorrelation $R(\tau) = E[X(t)X(t+\tau)]$ depends only on the time difference.

Note that stationarity implies wide-sense stationarity. However, the converse is not true: WSS does not imply stationarity.

Ergodic Processes

A process is said to be ***Ergodic*** if all orders of statistical and time averages are interchangeable. The mean, autocorrelation and other statistics can be computed by using any sample function of the process. That is

$$\begin{aligned} \eta = E[X(t)] &= \int_{-\infty}^{\infty} x\, f(x) dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\, dt \\ R(\tau) = E[X(t)X(t+\tau)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1, x_2; \tau) dx_1 dx_2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\, dt\,. \end{aligned} \tag{6-13}$$

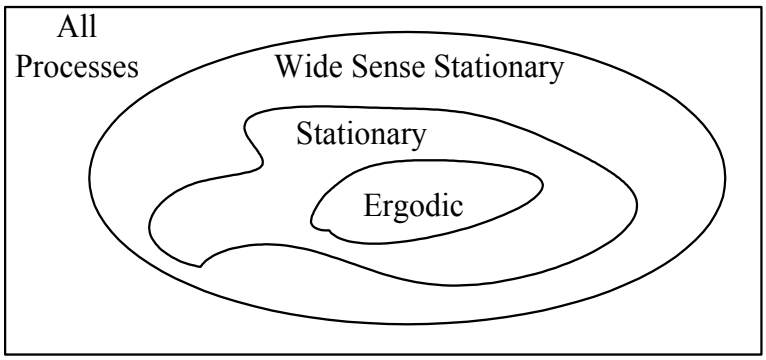

Figure 6-2: Hierarchy of random processes.

This idea extends to higher-order averages as well. Since we are averaging over absolute time, the ensemble averages (all orders) cannot depend on absolute time. This requires that the original process must be stationary. That is, ergodicity implies stationarity. However, the converse is not true: there are stationary processes that are not ergodic. The hierarchy of random processes is abstractly illustrated by Figure 6-2.

Example 6-4: Let X(t) = A, where A is uniformly distributed in the interval [0, 1]. Sample functions of X are straight lines, as shown by Figure 6-3. Clearly, X(t) is not ergodic since the time average of each sample function is different.

Example 6-5: Random Walk

The ***random walk*** is the quintessential example of a ***Markov process***, a type of process that has many applications in engineering and the physical sciences. Many versions of the

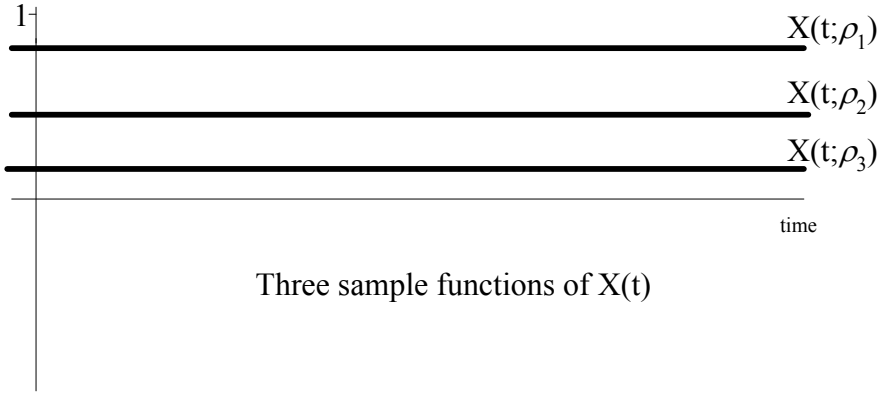

Figure 6-3: Sample functions of a non-ergodic random process.

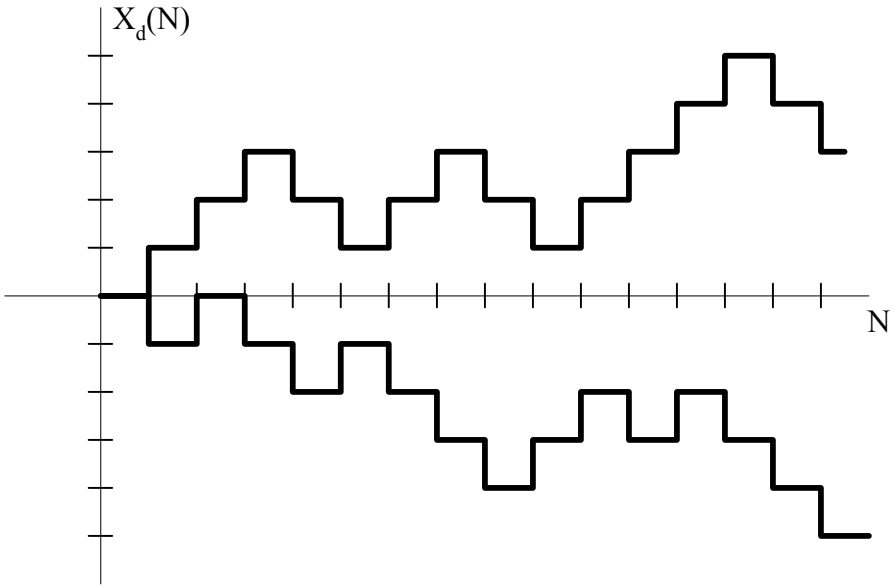

Figure 6-4: Two sample functions of a random walk process.

random walk have been studied over the years (*i.e.*, the ***gambler's ruin***, ***drunken sailor***, etc.). At first, a discrete random walk is introduced. Then, it is shown that a limiting form of the random walk is the well-known continuous Wiener process. Finally, simple equations are developed that provide a complete statistical description of the discrete and limiting form of the random walk.

Suppose a man takes a random walk by starting at a designated origin on a straight line path. With probability p (alternatively, $q \equiv 1 - p$), he takes a step to the right (alternatively, left). Suppose that each step is of length ℓ meters, and each step is completed in τ_s seconds. After N steps (completed in $N\tau_s$ seconds), the man is located $X_d(N)$ steps from the origin; note that $-N \le X_d(N) \le N$ since the man starts at the origin. If $X_d(N)$ is positive (alternatively, negative), the man is located to the right (alternatively, left) of the origin. The quantity $\mathbf{P}[X_d(N) = n]$, $-N \le n \le N$, denotes the probability that the man's location is n steps from the origin after he has taken N steps. Figure 6-4 depicts two sample functions of $X_d(N)$, a discrete random process.

The calculation of $\mathbf{P}[X_d(N) = n]$ is simplified greatly by the assumption, implied in the previous paragraph, that the man takes independent steps. That is, the direction taken at the N^{th} step is independent of $X_d(k)$, $0 \le k \le N - 1$, and the directions taken at all previous steps. Also

simplifying the development is the assumption that p does not depend on step index N. Under these conditions, it is possible to write

$$\begin{aligned}&\mathbf{P}[X_d(N+1) = X_d(N) + 1] = p\\ &\mathbf{P}[X_d(N+1) = X_d(N) - 1] = 1 - p = q\,.\end{aligned} \tag{6-14}$$

Let R_{n0} and L_{n0} denote the number of steps to the right and left, respectively, that will place the man n, $-N \le n \le N$, steps from the origin after he has completed a total of N steps. Integers R_{n0} and L_{n0} depend on integers N and n; the relationship is given by

$$\begin{aligned}&R_{n0} - L_{n0} = n\\ &R_{n0} + L_{n0} = N\end{aligned} \tag{6-15}$$

since $-N \le n \le N$. Integer values for R_{n0} and L_{n0} exist only if $-N \le n \le N$ and the pair of integers (N + n), (N - n) are even. When an integer solution of (6-15) exists it is given by

$$\begin{aligned}&R_{n0} = \frac{N + n}{2}\\ &L_{n0} = \frac{N - n}{2}\end{aligned}\,, \tag{6-16}$$

for $-N \le n \le N$, and (N+n), (N-n) even integers. After taking a total of N steps, it is not possible to reach n steps (to the right of the origin) if integer values for R_{n0} and L_{n0} do not exist. Also, if it is possible to reach n steps (to the right of the origin after taking a total of N steps), then it is not possible to reach n ± 1 steps (to the right of the origin).

Of course, there are multiple sequences of N steps, R_{n0} to the right and L_{n0} to the left, that the man can take to insure that he is n steps to the right of the origin. In fact, the number of such sequences is given by

$$\binom{N}{R_{n0}} = \frac{N!}{R_{n0}!\,L_{n0}!}. \tag{6-17}$$

This quantity represents the number of subsets of size R_{n0} that can be formed from N distinct objects. These sequences are mutually exclusive events. Furthermore, they are equally probable, and the probability of each of them is

$$\mathbf{P}[R_{n0}\ (L_{n0})\ \text{Steps To Right (Left) in a Specific Sequence}] = p^{R_{n0}} q^{L_{n0}}. \tag{6-18}$$

The desired probability $\mathbf{P}[X_d(N) = n]$ can be computed easily with the use of (6-16), (6-17) and (6-18). From the theory of independent Bernoulli trials, the result

$$\mathbf{P}[X_d(N) = n] = \begin{cases} \dfrac{N!}{R_{n0}!\,L_{n0}!} p^{R_{n0}} q^{L_{n0}}, & \text{if integers } R_{n0} \text{ and } L_{n0} \text{ exit} \\ 0, & \text{if integers } R_{n0} \text{ and } L_{n0} \text{ do not exit} \end{cases} \tag{6-19}$$

follows easily. If there are no integer solutions to (6-15) for given values of n and N (*i.e.*, integers R_{n0} and L_{n0} do not exist), then it is not possible to arrive at n steps from the origin after taking N steps and $\mathbf{P}[X_d(N) = n] = 0$. Note that (6-19) is just the probability that the man takes R_{n0} steps to the right given that he takes N independent steps.

The analysis leading to (6-19) can be generalized to include a non-zero starting location. Instead of starting at the origin, assume that the man starts his random walk at m steps to the right of the origin. Then, after the man has completed N independent steps, $\mathbf{P}[X_d(N) = n \,|\, X_d(0) = m]$ denotes the probability that he is n steps to the right of the origin given that he started m steps to the right of the origin. A formula for $\mathbf{P}[X_d(N) = n \,|\, X_d(0) = m]$ is developed in what follows.

Let $\nu \equiv n - m$. The quantity ν denotes the man's net increase in the number of steps to

the right after he has completed N steps. Also, R_{nm} (alternatively, L_{nm}) denotes the number of steps to the right (alternatively, left) that are required if the man starts and finishes m and n, respectively, steps to the right of the origin. Note that $R_{nm} + L_{nm} = N$ and $R_{nm} - L_{nm} = \nu$ so that

$$R_{nm} = \frac{N + \nu}{2}$$
$$L_{nm} = \frac{N - \nu}{2}. \tag{6-20}$$

Solution (6-20) is valid only if $|\nu| \le N$ and integers $(N + \nu)$, $(N - \nu)$ are even. Otherwise, integers R_{nm} and L_{nm} do not exist, and it is not possible to start at m (steps to the right of the origin), take N independent steps, and find yourself at n (steps to the right of the origin). Finally, suppose that integers R_{nm} and L_{nm} exist for some n and m; that is, it is possible to go from m to n (steps to the right of the origin) in a total of N steps. Then, it is not possible to go from m to n ± 1 steps in a total of N steps.

The desired result follows by substituting R_{nm} and L_{nm} for R_{n0} and L_{n0} in (6-19); this procedure leads to

$$\mathbf{P}[X_d(N) = n \,|\, X_d(0) = m] = \mathbf{P}[R_{nm} \text{ steps to the right out of } N \text{ steps}]$$
$$= \frac{N!}{R_{nm}!(N - R_{nm})!} p^{R_{nm}} q^{L_{nm}} \tag{6-21}$$
$$= \binom{N}{R_{nm}} p^{R_{nm}} q^{L_{nm}}$$

if integers R_{nm} and L_{nm} exist, and

$$\mathbf{P}[X_d(N) = n \,|\, X_d(0) = m] = 0 \tag{6-22}$$

if R_{nm} and L_{nm} do not exist.

To simplify the developments in the remainder of this chapter, it is assumed that p = q = 1/2. Also, R_{nm} is assumed to exist in what follows. Otherwise, results (that use R_{nm}) given below can be modified easily if n and m are such that R_{nm} does not exist.

Process $X_d(N)$ has ***independent increments***. That is, consider integers N_1, N_2, N_3 and N_4, where $N_1 < N_2 \le N_3 < N_4$. Then $X_d(N_2) - X_d(N_1)$ is statistically independent of $X_d(N_4) - X_d(N_3)$.

The Wiener Process As a Limit of the Random Walk

Recall that each step corresponds to a distance of ℓ meters, and each step is completed in τ_s seconds. At time $t = N\tau_s$, let $X(N\tau_s)$ denote the man's physical displacement (in meters) from the origin. Then $X(N\tau_s)$ is a random process given by $X(N\tau_s) \equiv \ell X_d(N)$, since $X_d(N)$ denotes the number of steps the man is from the origin after he takes N steps. Note that $X(N\tau_s)$ is a discrete-time random process that takes on only discrete values.

For large N and small ℓ and τ_s, the probabilistic nature of $X(N\tau_s)$ is of interest. First, note that $\mathbf{P}[X(N\tau_s) = \ell n \,|\, X(0) = \ell m] = \mathbf{P}[X_d(N) = n \,|\, X_d(0) = m]$; this observation and the Binomial distribution function leads to the result (use p = q = ½)

$$\mathbf{P}[X(N\tau_s) \le \ell n \,|\, X(0) = \ell m] = \mathbf{P}[X_d(N) \le n \,|\, X_d(0) = m]$$

$$= \mathbf{P}[\text{from N steps, the number k taken to right is} \le R_{nm}] \quad (6\text{-}23)$$

$$= \sum_{k=0}^{R_{nm}} \binom{N}{k} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{N-k}.$$

For large N, the DeMoivre-Laplace theorem (see Chapter 1 of these class notes) leads to the approximation

$$\mathbf{P}[X(N\tau_s) \le \ell n \mid X(0) = \ell m] \approx \mathcal{G}\left(\frac{R_{nm} - N/2}{\sqrt{N/4}}\right) = \mathcal{G}\left(\frac{\nu}{\sqrt{N}}\right),$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\nu/\sqrt{N}} \exp\left[-\tfrac{1}{2}u^2\right] du \qquad (6\text{-}24)$$

where $\mathcal{G}$ is the distribution function for a zero-mean, unit-variance Gaussian random variable.

The discrete random walk process outlined above has a continuous process as a formal limit. To see this, let $\ell \to 0$, $\tau_s \to 0$ and $N \to \infty$ in such a manner that

$$\frac{\ell^2}{2\tau_s} \to D$$
$$t = N\tau_s$$
$$x = n\ell \qquad (6\text{-}25)$$
$$x_0 = m\ell$$
$$X(t) = X(N\tau_s),$$

where D is known as the ***diffusion constant***. In terms of D, x, x_0 and t, the results of (6-25) can be used to write

$$\frac{\nu}{\sqrt{N}} = \frac{(x - x_0)/\ell}{\sqrt{t/\tau_s}} = \frac{(x - x_0)}{\sqrt{2Dt}}. \qquad (6\text{-}26)$$

Process $X(t)$ is a continuous random process.

The probabilistic nature of the limiting form of $X(t)$ is seen from (6-24) and (6-26). In the limit, the process $X(t)$ is described by the first-order conditional distribution function

$$F(x;t|x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-x_0)/\sqrt{2Dt}} \exp\left[-\tfrac{1}{2}u^2\right] du\,, \qquad (6\text{-}27)$$

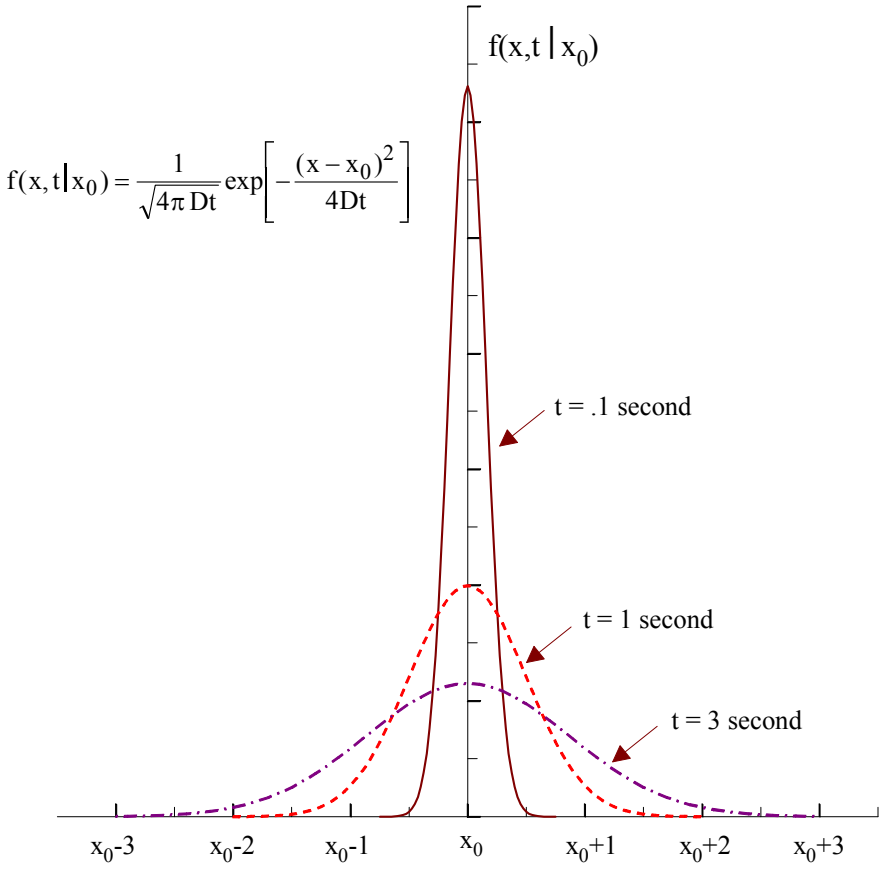

Figure 6-5: Density function for a diffusion process with D = 1.

and the first-order conditional density function

$$f(x,t\,|\,x_0) = \frac{d}{dx}F(x;t|x_0) = \frac{1}{\sqrt{4\pi\,Dt}}\exp\left[-\frac{(x-x_0)^2}{4\,Dt}\right], \tag{6-28}$$

a result depicted by Figure 6-5. Often, $f(x,t\,|\,x_0)$ is known as the ***transition density function*** for the process since $f(x,t\,|\,x_0)\Delta x$ is the probability of making the transition from x_0 to the interval $(x, x+\Delta x)$ by time t. Equation (6-28) describes the conditional probability density function of a continuous-time ***Wiener process***. Clearly, process $X(t)$ is Gaussian, it has a mean of x_0, and it has a variance that grows with time (hence, it is nonstationary). Finally, as $t \to 0^+$, note that $f(x,t\,|\,x_0) \to \delta(x - x_0)$, as expected.

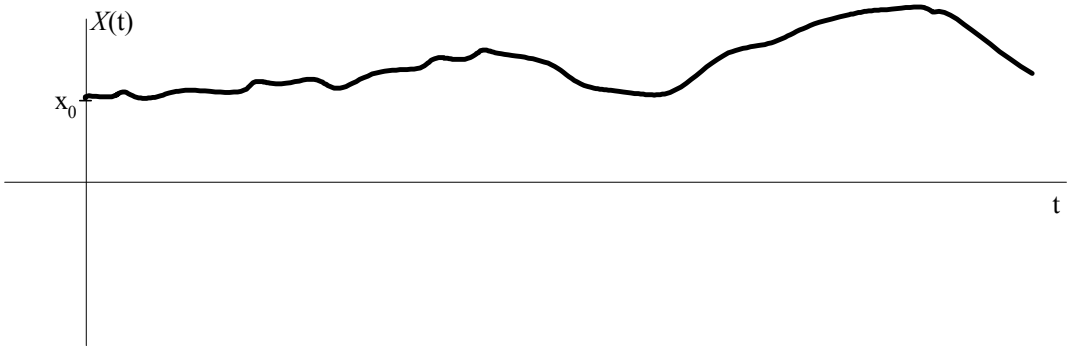

Figure 6-6: A hypothetical sample function of a Wiener process $X(t)$.

Figure 6-6 attempts to depict a sample function of a Wiener process. While such drawings are "nice" to look at (and they are fun to draw!), they cannot depict accurately all attributes of a Wiener process sample function. As it turns out, Wiener processes have continuous (in time) sample functions; there won't be a step or jump in the sample function of a Wiener process. However, in the traditional Calculus sense, the sample functions are differentiable nowhere. That is, in the classical Calculus sense, the derivative dX/dt does not exist at any value of time (actually, it's just a little more complicated than this). A generalized derivative of the Wiener process does exit, however. In engineering and the physical sciences, it is known as white Gaussian noise.

Process $X(t)$ has ***independent increments***. Let (t_1, t_2), (t_3, t_4) be non-overlapping intervals ($t_1 < t_2 \le t_3 < t_4$). Then increment $X(t_2) - X(t_1)$ is independent of increment $X(t_4) - X(t_3)$. Finally, increment $X(t_2) - X(t_1)$ has a mean of zero and a variance of $2D|t_2 - t_1|$.

The Diffusion Equation for the Transition Density Function

In terms of physical displacement (from the origin) X, the conditional probability $\mathbf{P}[X(N\tau_s) = \ell n \,|\, X(0) = \ell m]$ describes the probabilistic nature of the discrete time random walk problem outlined above. In what follows, this conditional probability is denoted by the short hand notation $\mathbf{P}[\ell n, N\tau_s \,|\, \ell m]$. For the case p = q = 1/2, it is easy to see that it satisfies the difference equation

$$\mathbf{P}[\ell n, (N+1)\tau_s \,|\, \ell m] = \tfrac{1}{2}\mathbf{P}[\ell(n-1), N\tau_s \,|\, \ell m] + \tfrac{1}{2}\mathbf{P}[\ell(n+1), N\tau_s \,|\, \ell m]. \tag{6-29}$$

That is, to get to ℓn at time $(N+1)\tau_s$, you can be at $\ell(n-1)$ at time $N\tau_s$ and take a step to the right (this occurs with probability equal to 1/2), or you can be at $\ell(n+1)$ at time $N\tau_s$ and take a step to the left (this occurs with probability equal to 1/2). Equation (6-29) can be applied twice to obtain

$$\begin{aligned}\mathbf{P}[\ell n, (N+2)\tau_s \mid \ell m] &= \tfrac{1}{2}\mathbf{P}[\ell(n-1), (N+1)\tau_s \mid \ell m] + \tfrac{1}{2}\mathbf{P}[\ell(n+1), (N+1)\tau_s \mid \ell m] \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}\mathbf{P}[\ell(n-2), N\tau_s \mid \ell m] + \tfrac{1}{2}\mathbf{P}[\ell n, N\tau_s \mid \ell m]\right] \\ &\quad + \tfrac{1}{2}\left[\tfrac{1}{2}\mathbf{P}[\ell n, N\tau_s \mid \ell m] + \tfrac{1}{2}\mathbf{P}[\ell(n+2), N\tau_s \mid \ell m]\right]\end{aligned} \tag{6-30}$$

This last result can be simplified to obtain

$$\begin{aligned}&\mathbf{P}[\ell n, (N+2)\tau_s \mid \ell m] \\ &= \tfrac{1}{4}\mathbf{P}[\ell(n-2), N\tau_s \mid \ell m] + \tfrac{1}{2}\mathbf{P}[\ell n, N\tau_s \mid \ell m] + \tfrac{1}{4}\mathbf{P}[\ell(n+2), N\tau_s \mid \ell m]\end{aligned} \tag{6-31}$$

The continuous conditional density $f(x,t \mid x_0)$ given by (6-28) satisfies a partial differential equation. To obtain this equation, first note that the difference equation (6-31) can be used to write

$$\begin{aligned}&\frac{\mathbf{P}[\ell n, (N+2)\tau_s \mid \ell m] - \mathbf{P}[\ell n, N\tau_s \mid \ell m]}{2\tau_s} \\ &= \frac{\ell^2}{2\tau_s}\left[\frac{\mathbf{P}[\ell(n+2), N\tau_s \mid \ell m] - 2\mathbf{P}[\ell n, N\tau_s \mid \ell m] + \mathbf{P}[\ell(n-2), N\tau_s \mid \ell m]}{(2\ell)^2}\right]\end{aligned} \tag{6-32}$$

(substitute (6-31) for the first term on the left-hand side of (6-32) to verify the expression). Now, in the sense described by (6-25), the formal limit of (6-32) can be found. To find this limiting form, we must consider two results. First, note that (6-24) and (6-25) imply

$$\mathbf{P}[\ell n, N\tau_s \mid \ell m] = \mathbf{P}[X(N\tau_s) \le \ell n \mid X(0) = \ell m] - \mathbf{P}[X(N\tau_s) \le \ell(n-2) \mid X(0) = \ell m]$$

$$\approx \mathcal{G}\left(\frac{\nu}{\sqrt{N}}\right) - \mathcal{G}\left(\frac{\nu-2}{\sqrt{N}}\right) \approx \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\nu}{\sqrt{N}}\right)^2\right] \frac{2}{\sqrt{N}}$$

$$= \frac{2\ell}{\sqrt{4\pi\left(\frac{\ell^2}{2\tau_s}\right)N\tau_s}} \exp\left[-\frac{1}{2}\left(\frac{\nu}{\sqrt{N}}\right)^2\right] \quad . \tag{6-33}$$

$$= \frac{2\ell}{\sqrt{4\pi D t}} \exp\left[-\frac{1}{2}\frac{(x-x_0)^2}{2Dt}\right] = 2\ell\, f(x;t \mid x_0)$$

This last equation shows that, in the limit describe by (6-25), $\mathbf{P}[\ell n, N\tau_s \mid \ell m]$ approaches $2\ell f(x;t \mid x0)$. That is, as $\ell \to 0$, $\tau_s \to 0$, $\ell^2/2\tau_s \to D$ as described by (6-25), we know that $\mathbf{P}[\ell n, N\tau_s \mid \ell m]$ approaches zero according to

$$\mathbf{P}[\ell n, N\tau_s \mid \ell m] \to 2\ell\, f(x;t \mid x_0) \to 0\,. \tag{6-34}$$

Second, we must review expressions for derivatives; let g(x) be an ordinary function of x, and recall that the first partial derivative of g can be expressed as

$$\frac{\partial}{\partial x} g(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x-\Delta x)}{2\Delta x}\,. \tag{6-35}$$

Use this formula to express the second derivative of g as

$$\frac{\partial^2}{\partial x^2} g(x) = \lim_{\Delta x \to 0} \frac{[g(x+2\Delta x) - g(x)] - [g(x) - g(x-2\Delta x)]}{(2\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{g(x+2\Delta x) - 2g(x) + g(x-2\Delta x)}{(2\Delta x)^2} \tag{6-36}$$

From (6-34), (6-35) and (6-36), the formal limit of (6-32) is

$$\frac{\partial}{\partial t} f(x,t \mid x_0) = D \frac{\partial^2}{\partial x^2} f(x,t \mid x_0), \tag{6-37}$$

where $f(x,t \mid x_0)$ denotes the conditional probability density function given by (6-28). Note that Equation (6-37) is identical in form to the source-free, one-dimensional heat equation. Probability diffuses just like heat and electronic charge (and many other physical phenomenon)!

Equation (6-37) is a one-dimensional ***diffusion equation***. It describes how probability diffuses (or flows) with time. It implies that probability is conserved in much the same way that the well know continuity equation implies the conservation of electric charge. To draw this analogy, note that f describes the density of probability (or density of probability particles) on the one-dimensional real line. That is, f can be assigned units of particles/meter. Since D has units of meters^2/second, a units check on both sides of (6-37) produces

$$\left(\frac{1}{\text{second}}\right)\left(\frac{\text{particles}}{\text{meter}}\right) = \left(\frac{\text{meter}^2}{\text{second}}\right)\left(\frac{1}{\text{meter}}\right)^2\left(\frac{\text{particles}}{\text{meter}}\right). \tag{6-38}$$

Now, write (6-37) as

$$\frac{\partial}{\partial t} f = -\nabla \cdot \Im, \tag{6-39}$$

where

$$\Im \equiv -D \frac{\partial}{\partial x} f, \tag{6-40}$$

and ∇ is the divergence operator. The quantity $\Im$ is a one-dimensional ***probability current***, and

it has units of particles/second. Note the similarity between (6-39) and the well-known continuity equation for electrical charge.

Probability current $\Im(x,t\,|\,x_0)$ indicates the rate of particle flow past point x at time t. Let (x_1, x_2) denote an interval; integrate (6-39) over this interval to obtain

$$\frac{\partial}{\partial t}\mathbf{P}[x_1 < X(t) \le x_2\,|\,x_0] = \frac{\partial}{\partial t}\int_{x_1}^{x_2} f(x,t\,|\,x_0)\,dx = -\Im(x_2,t\,|\,x_0) + \Im(x_1,t\,|\,x_0)]. \tag{6-41}$$

As illustrated by Figure 6-7, the left-hand side of this equation represents the time rate of probability build-up on (x_1, x_2). That is, between the limits of x_1 and x_2, the area under f is changing at a rate equal to the left-hand side of (6-41). As depicted, the right-hand side of (6-41) represents the probability currents entering the ends of the interval (x_1, x_2).

The Wiener process is a simple example of a ***diffusion process***. Diffusion processes are important in the study of communication and control systems. As it turns out, the state vector that describes a system (such as a circuit, PLL, spring-mass, etc.) driven by white Gaussian noise is a diffusion process. Also, this state vector is described statistically by a density function that satisfies a partial differential equation known as the ***Fokker-Planck equation*** (of which (6-37) is a simple example). Finally, it should be pointed out that a diffusion process is a special case of a ***Markov process,*** a more general process (see Papoulis for the definition of a Markov process).

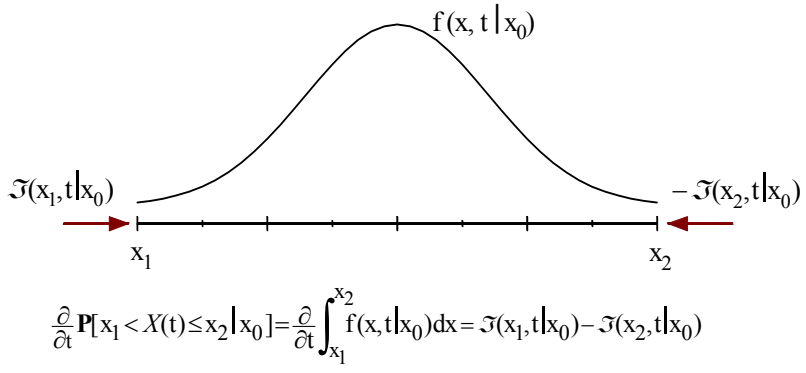

Figure 6-7: Probability build-up on the interval x_1, x_2 due to probability current entering the ends of the interval.

Solution of Diffusion Equation by Transform Techniques

The one-dimensional diffusion equation (6-37) can be solved by using transform techniques. First, initial and boundary conditions must be specified. The desired initial condition is

$$f(x,t|x_0)\Big|_{t=0} = \delta(x-x_0), \tag{6-42}$$

which means that random process x starts at x_0. What are known as *natural boundary conditions* are to be used; that is, we require

$$f(x,t|x_0)\Big|_{x=\infty} = 0. \tag{6-43}$$

Consider the transform of $f(x,t\,|\,x_0)$ defined by

$$\varphi(s,t) = \int_{-\infty}^{\infty} f(x,t|x_0)e^{jxs}\,dx. \tag{6-44}$$

With respect to t, differentiate (6-44) to obtain

$$\frac{\partial\varphi(s,t)}{\partial t} = \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial t}f(x,t|x_0)\right\}e^{jxs}\,dx = D\int_{-\infty}^{\infty}\left\{\frac{\partial^2}{\partial x^2}f(x,t|x_0)\right\}e^{jxs}\,dx. \tag{6-45}$$

Now, use

$$\begin{array}{l}\lim\limits_{x\to\infty} f(x,t|x_0) = 0 \\ \lim\limits_{x\to\infty}\dfrac{\partial f(x,t|x_0)}{\partial x} = 0\end{array}, \tag{6-46}$$

and integrate by parts twice to obtain

$$\frac{\partial \varphi}{\partial t} = -Ds^2\varphi \,. \tag{6-47}$$

This equation can be solved easily to obtain

$$\varphi(s,t) = \exp\left[-Ds^2 t\right]\varphi(s,0) \,. \tag{6-48}$$

However, from the initial condition (6-42), we have

$$\varphi(s,0) = \exp(jx_0 s) \,. \tag{6-49}$$

Finally, combine (6-48) and (6-49) to obtain

$$\varphi(s,t) = \exp\left[jx_0 s - Ds^2 t\right] . \tag{6-50}$$

But, this is the well-known characteristic function of the Gaussian density function

$$f(x,t \mid x_0) = \frac{1}{\sqrt{4\pi\, Dt}} \exp\left[-\frac{(x-x_0)^2}{4\,Dt}\right] . \tag{6-51}$$

This same technique can be used to solve higher-order, more complicated diffusion equations.