Chapter 7 - Correlation Functions

Let X(t) denote a random process. The *autocorrelation* of X is defined as

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;t_1,t_2)\,dx_1 dx_2\,. \tag{7-1}$$

The *autocovariance* function is defined as

$$C_X(t_1,t_2) = E[\{X(t_1)-\eta_X(t_1)\}\{X(t_2)-\eta_X(t_2)\}] = R_X(t_1,t_2) - \eta_X(t_1)\eta_X(t_2)\,, \tag{7-2}$$

and the *correlation function* is defined as

$$r_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{R_X(t_1,t_2)-\eta_X(t_1)\eta_X(t_2)}{\sigma_X(t_1)\sigma_X(t_2)}\,. \tag{7-3}$$

If X(t) is at least wide sense stationary, then R_X depends only on the time difference $\tau = t_1 - t_2$, and we write

$$R_X(\tau) = E[X(t)X(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1,x_2;\tau)\,dx_1 dx_2\,. \tag{7-4}$$

Finally, if X(t) is ergodic we can write

$$R_X(\tau) = \underset{T\to\infty}{\text{limit}}\,\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt\,. \tag{7-5}$$

Function $r_X(\tau)$ can be thought of as a "measure of statistical similarity" of X(t) and X(t+τ). If $r_x(\tau_0) = 0$, the samples X(t) and $X(t+\tau_0)$ are said to be *uncorrelated*.

Properties of Autocorrelation Functions for Real-Valued, WSS Random Processes

1. $R_X(0) = E[X(t)X(t)]$ = Average Power.

2. $R_X(\tau) = R_X(-\tau)$. The autocorrelation function of a real-valued, WSS process is even.

Proof:

$$\begin{aligned} R_X(\tau) &= E[X(t)X(t+\tau)] = E[X(t-\tau)X(t-\tau+\tau)] \text{ (Due to WSS)} \\ &= R_X(-\tau) \end{aligned} \tag{7-6}$$

3. $|R_X(\tau)| \le R_X(0)$. The autocorrelation is maximum at the origin.

Proof:

$$\begin{aligned} E\Big[\big(X(t) \pm X(t+\tau)\big)^2\Big] &= E\Big[X(t)^2 + X(t+\tau)^2 \pm 2X(t)X(t+\tau)\Big] \ge 0 \\ &= R_X(0) + R_X(0) \pm 2R_X(\tau) \ge 0 \end{aligned} \tag{7-7}$$

Hence, $|R_X(\tau)| \le R_X(0)$ as claimed.

4. Assume that WSS process X can be represented as $X(t) = \eta + X_{ac}(t)$, where η is a constant and $E[X_{ac}(t)] = 0$. Then,

$$\begin{aligned} R_X(\tau) &= E\big[\big(\eta + X_{ac}(t)\big)\big(\eta + X_{ac}(t+\tau)\big)\big] \\ &= E[\eta^2] + 2\eta E[X_{ac}(t)] + E\big[X_{ac}(t)X_{ac}(t+\tau)\big] \\ &= \eta^2 + R_{X_{ac}}(\tau). \end{aligned} \tag{7-8}$$

5. If each sample function of X(t) has a periodic component of frequency ω then $R_X(\tau)$ will have a periodic component of frequency ω.

Example 7-1: Consider X(t) = Acos(ωt+θ) + N(t), where A and ω are constants, random variable θ is uniformly distributed over (0, 2π), and wide sense stationary N(t) is independent of θ for every time t. Find $R_X(\tau)$, the autocorrelation of X(t).

$$R_X(\tau) = E\Big[\{A\cos(\omega t+\theta) + N(t)\}\{A\cos(\omega[t+\tau]+\theta) + N(t+\tau)\}\Big]$$

$$\begin{aligned} &= \frac{A^2}{2}E\big[\cos(2\omega t+\omega\tau+2\theta)+\cos(\omega\tau)\big] + E\big[A\cos(\omega t+\theta)N(t+\tau)\big] \\ &\qquad + E\big[N(t)A\cos(\omega[t+\tau]+\theta)\big] + E\big[N(t)N(t+\tau)\big] \end{aligned} \tag{7-9}$$

$$= \frac{A^2}{2}\cos(\omega\tau) + R_N(\tau)$$

So, $R_X(\tau)$ contains a component at ω, the same frequency as the periodic component in X.

6. Suppose that X(t) is ergodic, has zero mean, and it has no periodic components; then

$$\underset{\tau\to\infty}{\text{limit}}\, R_X(\tau) = 0\,. \tag{7-10}$$

That is, X(t) and X(t+τ) become uncorrelated for large τ.

7. Autocorrelation functions cannot have an arbitrary shape. As will be discussed in Chapter 8, for a WSS random process X(t) with autocorrelation $R_X(\tau)$, the Fourier transform of $R_X(\tau)$ is the *power density spectrum* (or simply *power spectrum*) of the random process X. And, the power spectrum must be non-negative. Hence, we have the additional requirement that

$$\mathcal{F}\big[R_X(\tau)\big] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} R_X(\tau)\cos(\omega\tau)d\tau \ge 0 \tag{7-11}$$

for all ω (the even nature of R_X was used to obtain the right-hand side of (7-11)). Because of this, in applications, you will not find autocorrelation functions with flat tops, vertical sides, or any jump discontinuities in amplitude (these features cause "oscillatory behavior", and negative values, in the Fourier transform). Autocorrelation $R(\tau)$ must vary smoothly with τ.

Example 7-2: Random Binary Waveform

Process X(t) takes on only two values: ±A. Every t_a seconds a sample function of X either "toggles" value or it remains the same (positive constant t_a is known). Both possibilities are equally likely (*i.e.,* **P**["toggle"] = **P**["no toggle"] = 1/2). The possible transitions occur at times $t_0 + kt_a$, where k is an integer, $-\infty < k < \infty$. Time t_0 is a random variable that is uniformly distributed over $[0, t_a]$. Hence, given an arbitrary sample function from the ensemble, a "toggle" can occur anytime. Starting from $t = t_0$, sample functions are constant over intervals of length t_a, and the constant can change sign from one t_a interval to the next. The value of X(t) over one "t_a-interval" is independent of its value over any other "t_a-interval". Figure 7-1 depicts a typical sample function of the random binary waveform. Figure 7-2 is a timing diagram that illustrates the "t_a intervals". The algorithm used to generate the process is not changing with time. As a result, it is possible to argue that the process is stationary. Also, since +A and –A are equally likely values for X at any time t, it is obvious that X(t) has zero mean.

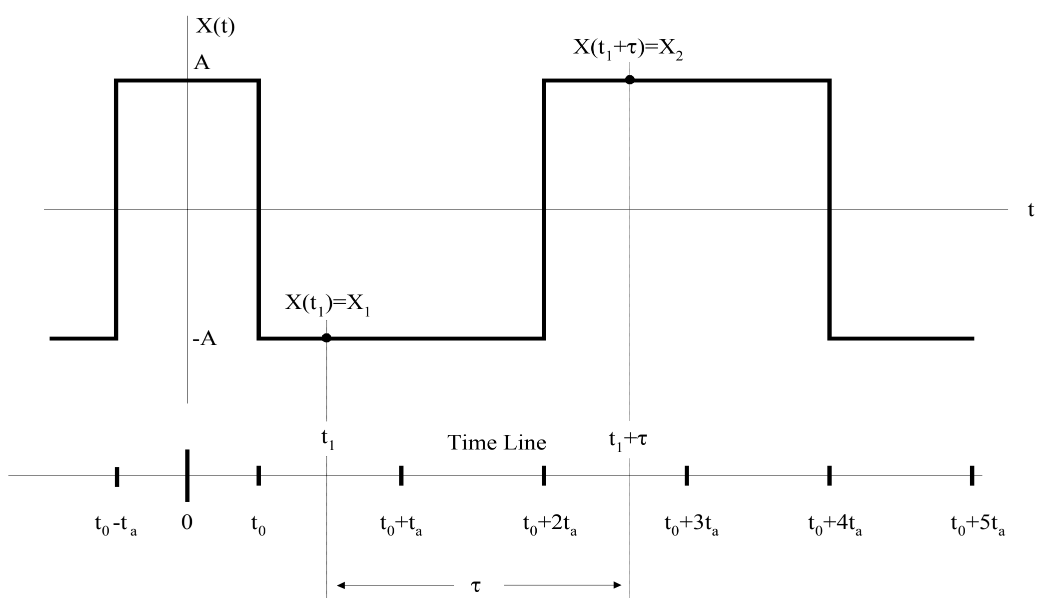

Fig. 7- 1: Sample function of a simple binary random process.

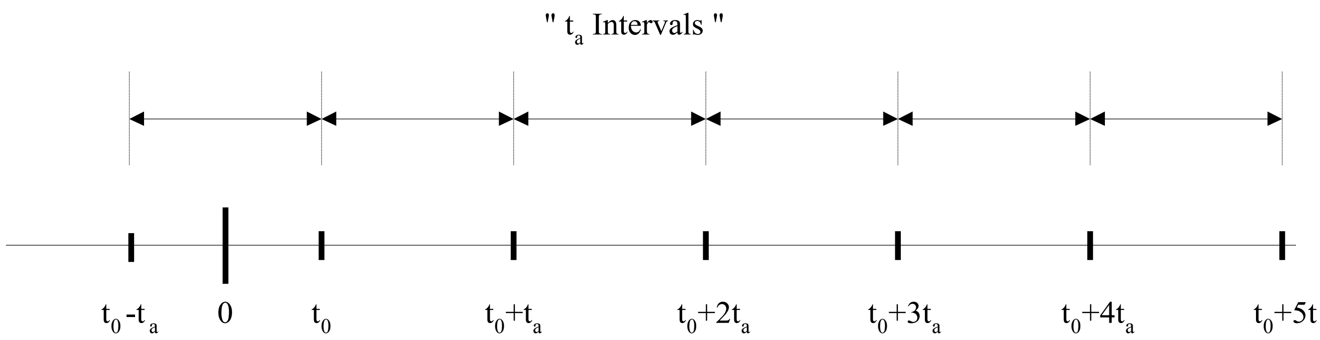

Fig. 7-2: Time line illustrating the independent "t_a intervals".

To determine the autocorrelation function $R_X(\tau)$, we must consider two basic cases.

1) Case $|\tau| > t_a$. Then, the times t_1 and $t_1 + \tau$ cannot be in the same "t_a interval". Hence, $X(t_1)$ and $X(t_1 + \tau)$ must be independent so that

$$R(\tau) = E[X(t_1)X(t_1+\tau)] = E[X(t_1)]E[X(t_1+\tau)] = 0, \ |\tau| > t_a . \tag{7-12}$$

2) Case $|\tau| < t_a$. To calculate $R(\tau)$ for this case, we must first determine an expression for the probability **P**[t_1 and $t_1 + \tau$ in the same "t_a interval"]. We do this in two parts: the first part is i) $0 < \tau < t_a$, and the second part is ii) $-t_a < \tau \le 0$.

i) $0 < \tau < t_a$. Times t_1 and $t_1+\tau$ may, or may not, be in the same "t_a-interval". However, we write

$$\begin{aligned}\mathbf{P}[t_1 \text{ and } t_1+\tau \text{ in same "}t_a \text{ interval"}] &= \mathbf{P}[t_0 \le t_1 \le t_1+\tau < t_0+t_a] \\ &= \mathbf{P}[t_1+\tau-t_a < t_0 \le t_1] \\ &= \frac{1}{t_a}[t_1-(t_1+\tau-t_a)] \\ &= \frac{t_a-\tau}{t_a}, \quad 0<\tau<t_a\end{aligned} \tag{7-13}$$

ii) $-t_a < \tau \le 0$. Times t_1 and $t_1+\tau$ may, or may not, be in the same "t_a-interval". However, we write

$$\begin{aligned}\mathbf{P}[t_1 \text{ and } t_1+\tau \text{ in same "}t_a \text{ interval"}] &= \mathbf{P}[t_0 \le t_1+\tau \le t_1 < t_0+t_a] \\ &= \mathbf{P}[t_1 - t_a < t_0 \le t_1+\tau] \\ &= \frac{1}{t_a}[t_1+\tau-(t_1-t_a)] \\ &= \frac{t_a+\tau}{t_a}\ , \quad -t_a < \tau \le 0\end{aligned} \tag{7-14}$$

Combine (7-13) and (7-14), we can write

$$\mathbf{P}[t_1 \text{ and } t_1+\tau \text{ in same "}t_a \text{ interval"}] = \frac{t_a-|\tau|}{t_a}\ , \quad |\tau| < t_a\,. \tag{7-15}$$

Now, the product $X(t_1)X(t_1+\tau)$ takes on only two values, plus or minus A^2. If t_1 and $t_1+\tau$ are in the same "t_a-interval" then $X(t_1)X(t_1+\tau) = A^2$. If t_1 and $t_1+\tau$ are in different "t_a-intervals" then $X(t_1)$ and $X(t_1+\tau)$ are independent, and $X(t_1)X(t_1+\tau) = \pm A^2$ equally likely. For $|\tau| < t_a$ we can write

$$\begin{aligned}R(\tau) &= E\big[X(t_1)X(t_1+\tau)\big] \\ &= A^2\mathbf{P}\big[t_1 \text{ and } t_1+\tau \text{ in same "}t_a \text{ interval"}\big] \\ &\quad +A^2\mathbf{P}\Big[\{t_1 \text{ and } t_1+\tau \text{ in different "}t_a \text{ intervals"}\},\ X(t_1)X(t_1+\tau)=A^2\Big] \\ &\quad -A^2\mathbf{P}\Big[\{t_1 \text{ and } t_1+\tau \text{ in different "}t_a \text{ intervals"}\},\ X(t_1)X(t_1+\tau)=-A^2\Big]\end{aligned} \tag{7-16}$$

However, the last two terms on the right-hand side of (7-16) cancel out (read again the two sentences after (7-15)). Hence, we can write

$$R(\tau) = A^2\mathbf{P}\big[t_1 \text{ and } t_1+\tau \text{ in same "}t_a \text{ interval"}\big]. \tag{7-17}$$

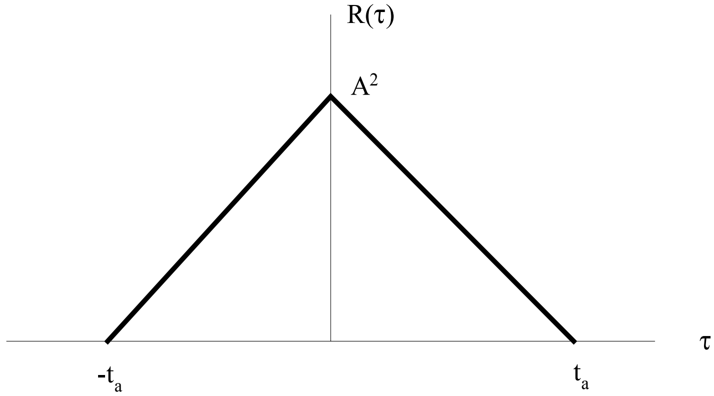

Fig. 7-3: Autocorrelation of Random Binary waveform.

Finally, substitute (7-15) into (7-17) to obtain

$$R(\tau) = E[X(t_1)X(t_1+\tau)] = \begin{cases} A^2\left[\dfrac{t_a - |\tau|}{t_a}\right], & |\tau| < t_a \\ 0, & |\tau| > t_a \end{cases} \quad . \tag{7-18}$$

Equation (7-18) provides a formula for $R(\tau)$ for the random binary signal described by Figure 7-1. A plot of this formula for $R(\tau)$ is given by Figure 7-3.

Poisson Random Points Review

The topic of random Poisson points is discussed in Chapters 1, 2 and Appendix 9B. Let $n(t_1, t_2)$ denote the number of Poisson points in the time interval (t_1, t_2). Then, these points are distributed in a Poisson manner with

$$\mathbf{P}[n(t_1,t_2) = k] = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \tag{7-19}$$

where $\tau \equiv |t_1 - t_2|$, and $\lambda > 0$ is a known parameter. That is, $n(t_1,t_2)$ is Poisson distributed with parameter $\lambda\tau$. Note that $n(t_1, t_2)$ is an integer valued random variable with

$$E[n(t_1, t_2)] = \lambda \left| t_1 - t_2 \right|$$

$$VAR[n(t_1, t_2)] = \lambda \left| t_1 - t_2 \right| \tag{7–20}$$

$$E[n^2(t_1, t_2)] = VAR[n(t_1, t_2)] + (E[n(t_1, t_2)])^2 = \lambda \left| t_1 - t_2 \right| + \lambda^2 \left| t_1 - t_2 \right|^2.$$

Note that $E[n(t_1,t_2)]$ and $VAR[n(t_1,t_2)]$ are the same, an unusual result for random quantities. If (t_1, t_2) and (t_3, t_4) are non-overlapping, then the random variables $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent. Finally, constant λ is the average point density. That is, λ represents the average number of points in a unit length interval.

Poisson Random Process

Define the *Poisson random process*

$$\begin{aligned} X(t) &= 0, \quad && t = 0 \\ &= n(0,t), \quad && t > 0 \end{aligned} \tag{7-21}$$

A typical sample function is illustrated by Figure 7-4.

Mean of Poisson Process

For any fixed $t \ge 0$, $X(t)$ is a Poisson random variable with parameter λt. Hence,

$$E[X(t)] = \lambda t, \;\; t \ge 0. \tag{7-22}$$

The time varying nature of the mean implies that process $X(t)$ is nonstationary.

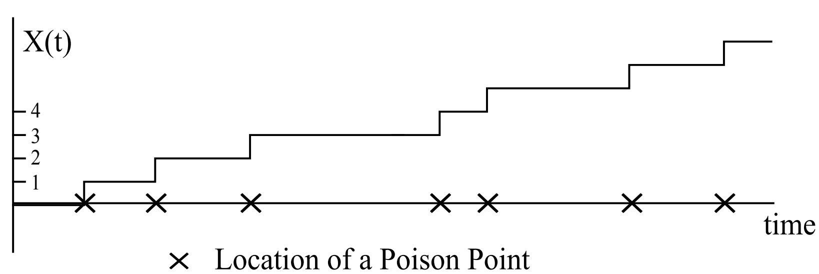

Fig. 7-4: Typical sample function of Poisson random process.

Autocorrelation of Poisson Process

The autocorrelation is defined as $R(t_1, t_2) = E[X(t_1)X(t_2)]$ for $t_1 \ge 0$ and $t_2 \ge 0$. First, note that

$$R(t,t) = \lambda t + \lambda^2 t^2 \,, \quad t > 0\,, \tag{7-23}$$

a result obtained from the known 2nd moment of a Poisson random variable. Next, we show that

$$\begin{aligned} R(t_1,t_2) &= \lambda t_2 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_2 < t_1 \\ &= \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_1 < t_2 \end{aligned} \,. \tag{7-24}$$

Proof: case $0 < t_1 < t_2$

We consider the case $0 < t_1 < t_2$. The random variables $X(t_1)$ and $\{X(t_2) - X(t_1)\}$ are independent since they are for non-overlapping time intervals. Also, $X(t_1)$ has mean λt_1, and $\{X(t_2) - X(t_1)\}$ has mean $\lambda(t_2 - t_1)$. As a result,

$$E[X(t_1)\{X(t_2) - X(t_1)\}] = E[X(t_1)]E[X(t_2) - X(t_1)] = \lambda t_1 \cdot \lambda(t_2 - t_1)\,. \tag{7-25}$$

Use this result to obtain

$$\begin{aligned} R(t_1,t_2) &= E[X(t_1)X(t_2)] = E[X(t_1)\{X(t_1) + X(t_2) - X(t_1)\}] \\ &= E[X^2(t_1)] + E[X(t_1)\{X(t_2) - X(t_1)\}] = \lambda t_1 + \lambda^2 t_1^2 + \lambda t_1 \cdot \lambda(t_2 - t_1)\,. \\ &= \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_1 < t_2 \end{aligned} \tag{7-26}$$

Case $0 < t_2 < t_1$ is similar to the case shown above. Hence, for the Poisson process, the autocorrelation function is

$$\begin{aligned} R(t_1,t_2) &= \lambda t_2 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_2 < t_1 \\ &= \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_1 < t_2 \end{aligned} \tag{7-27}$$

Semi-Random Telegraph Signal

The Semi-Random Telegraph Signal is defined as

$$\begin{aligned} X(0) &= 1 \\ X(t) &= 1 \text{ if number of Poisson Points in } (0,t) \text{ is } \textbf{\textit{even}} \\ &= -1 \text{ if number of Poisson Points in } (0,t) \text{ is } \textbf{\textit{odd}} \end{aligned} \tag{7-28}$$

for $-\infty < t < \infty$. Figure 7-5 depicts a typical sample function of this process. In what follows, we find the mean and autocorrelation of the semi-random telegraph signal.

First, note that

$$\begin{aligned} \mathbf{P}[X(t)=1] &= \mathbf{P}[\text{even number of pts in } (0,t)] \\ &= \mathbf{P}[0 \text{ pts in } (0,t)] + \mathbf{P}[2 \text{ pts in } (0,t)] + \mathbf{P}[4 \text{ pts in } (0,t)] + \cdots \\ &= e^{-\lambda|t|}\left(1 + \frac{\lambda^2|t|^2}{2!} + \frac{\lambda^4|t|^4}{4!} + \cdots\right) = e^{-\lambda|t|}\cosh(\lambda|t|)\,. \end{aligned} \tag{7-29}$$

Note that (7-29) is valid for $t < 0$ since it uses $|t|$. In a similar manner, we can write

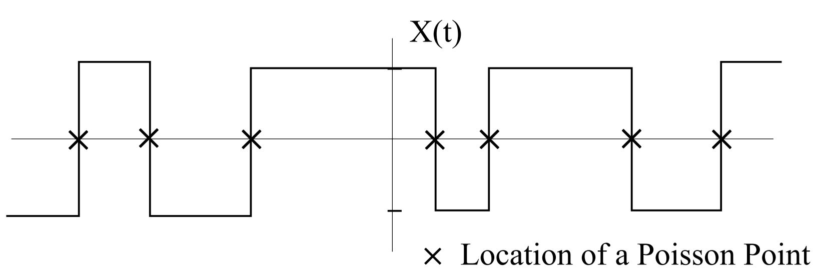

Fig. 7-5: A typical sample function of the semi-random telegraph signal.

$$\mathbf{P}[X(t) = -1] = \mathbf{P}[\text{odd number of pts in } (0, t)]$$

$$= \mathbf{P}[1 \text{ pts in } (0,t)] + \mathbf{P}[3 \text{ pts in } (0,t)] + \mathbf{P}[5 \text{ pts in } (0,t)] + \cdots \tag{7-30}$$

$$= e^{-\lambda|t|}\left(\lambda|t| + \frac{\lambda^3|t|^3}{3!} + \frac{\lambda^5|t|^5}{5!} + \cdots\right) = e^{-\lambda|t|}\sinh(\lambda|t|).$$

As a result of (7-29) and (7-30), the mean is

$$E[X(t)] = +1 \times \mathbf{P}[X(t) = +1] - 1 \times \mathbf{P}[X(t) = -1]$$

$$= e^{-\lambda|t|}\left(\cosh(\lambda|t|) - \sinh(\lambda|t|)\right) \tag{7-31}$$

$$= e^{-2\lambda|t|}.$$

The constraint X(0) = 1 causes a nonzero mean that dies out with time. Note that X(t) is **not** WSS since its mean is time varying.

Now, find the autocorrelation $R(t_1, t_2)$. First, suppose that $t_1 - t_2 \equiv \tau > 0$, and $-\infty < t_2 < \infty$. If there is an even number of points in (t_2, t_1), then $X(t_1)$ and $X(t_2)$ have the same sign and

$$\begin{aligned}\mathbf{P}[X(t_1) = 1,\ X(t_2) = 1] &= \mathbf{P}[\ X(t_1) = 1 \mid X(t_2) = 1\ ]\ \mathbf{P}[\ X(t_2) = 1\ ] \\ &= \{\exp[-\lambda\tau]\cosh(\lambda\tau)\}\,\{\exp[-\lambda|t_2|]\cosh(\lambda|t_2|)\}\end{aligned} \tag{7-32}$$

$$\begin{aligned}\mathbf{P}[X(t_1) = -1,\ X(t_2) = -1] &= \mathbf{P}[\ X(t_1) = -1 \mid X(t_2) = -1\ ]\ \mathbf{P}[\ X(t_2) = -1\ ] \\ &= \{\exp[-\lambda\tau]\cosh(\lambda\tau)\}\,\{\exp[-\lambda|t_2|]\sinh(\lambda|t_2|)\}\end{aligned} \tag{7-33}$$

for $t_1 - t_2 \equiv \tau > 0$, and $-\infty < t_2 < \infty$. If there are an odd number of points in (t_2, t_1), then $X(t_1)$ and $X(t_2)$ have different signs, and we have

$$\mathbf{P}[X(t_1)=1,\ X(t_2)=-1] = \mathbf{P}[\,X(t_1)=1 \,\big|\, X(t_2)=-1\,]\,\mathbf{P}[\,X(t_2)=-1\,]$$
$$= \{\exp[-\lambda\tau]\sinh(\lambda\tau)\}\{\exp[-\lambda|t_2|]\sinh(\lambda|t_2|)\} \tag{7-34}$$

$$\mathbf{P}[X(t_1)=-1,\ X(t_2)=1] = \mathbf{P}[\,X(t_1)=-1 \,\big|\, X(t_2)=1\,]\,\mathbf{P}[\,X(t_2)=1\,]$$
$$= \{\exp[-\lambda\tau]\sinh(\lambda\tau)\}\{\exp[-\lambda|t_2|]\cosh(\lambda|t_2|)\} \tag{7-35}$$

for $t_1 - t_2 \equiv \tau > 0$, and $-\infty < t_2 < \infty$. The product $X(t_1)X(t_2)$ is +1 with probability given by the sum of (7-32) and (7-33); it is -1 with probability given by the sum of (7-34) and (7-35). Hence, its expected value can be expressed as

$$\begin{aligned} R(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= e^{-\lambda\tau}\cosh(\lambda\tau)\left[e^{-\lambda|t_2|}\{\cosh(\lambda|t_2|)+\sinh(\lambda|t_2|)\}\right] \\ &\quad -e^{-\lambda\tau}\sinh(\lambda\tau)\left[e^{-\lambda|t_2|}\{\cosh(\lambda|t_2|)+\sinh(\lambda|t_2|)\}\right]. \end{aligned} \tag{7-36}$$

Using standard identities, this result can be simplified to produce

$$\begin{aligned} R(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= e^{-\lambda\tau}\left[\cosh(\lambda\tau)-\sinh(\lambda\tau)\right]e^{-\lambda|t_2|}\left[\cosh(\lambda|t_2|)+\sinh(\lambda|t_2|)\right] \\ &= e^{-\lambda\tau}\left[e^{-\lambda\tau}\right]e^{-\lambda|t_2|}\left[e^{\lambda|t_2|}\right] \\ &= e^{-2\lambda\tau}, \quad \text{for } \tau = t_1 - t_2 > 0. \end{aligned} \tag{7-37}$$

Due to symmetry (the autocorrelation function must be even), we can conclude that

$$R(t_1, t_2) = R(\tau) = e^{-2\lambda|\tau|}, \ \tau = |t_1 - t_2|, \tag{7-38}$$

is the autocorrelation function of the semi-random telegraph signal, a result illustrated by Fig. 7-6. Again, note that the semi-random telegraph signal is **not** WSS since it has a time-varying mean.

Random Telegraph Signal

Let X(t) denote the semi-random telegraph signal discussed above. Consider the process Y(t) = αX(t), where α is a random variable that is independent of X(t) for all time. Furthermore, assume that α takes on only two values: $\alpha = +1$ and $\alpha = -1$ equally likely. Then the mean of Y is E[Y] = E[αX] = E[α]E[X] = 0 for all time. Also, $R_Y(\tau) = E[\alpha^2]R_X(\tau) = R_X(\tau)$, a result depicted by Figure 7-6. Y is called the *Random Telegraph Signal* since it is "entirely random" for all time t. Note that the Random Telegraph Signal **is** WSS.

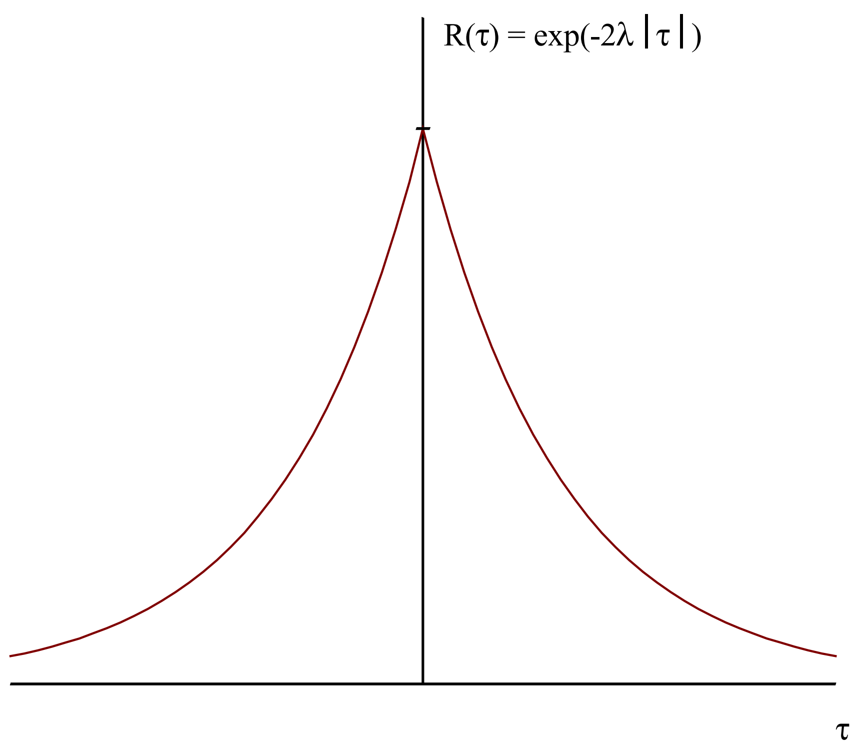

Fig. 7-6: Autocorrelation function for both the semi-random and random telegraph signals.

Autocorrelation of Wiener Process

Consider the Wiener process that was introduced in Chapter 6. If we assume that $X(0) = 0$ (in many textbooks, this is part of the definition of a Wiener process), then the autocorrelation of the Wiener process is $R_X(t_1, t_2) = 2D\{\min(t_1,t_2)\}$. To see this, first recall that a Wiener process has independent increments. That is, if (t_1, t_2) and (t_3, t_4) are non-overlapping intervals (*i.e.,* $0 \le t_1 < t_2 \le t_3 < t_4$), then increment $X(t_2) - X(t_1)$ is statistically independent of increment $X(t_4) - X(t_3)$. Now, consider the case $t_1 > t_2 \ge 0$ and write

$$\begin{aligned}R_X(t_1,t_2) &= E\big[X(t_1)X(t_2)\big] = E\big[\{X(t_1) - X(t_2) + X(t_2)\}X(t_2)\big]\\ &= E\big[\{X(t_1) - X(t_2)\}X(t_2)\big] + E\big[X(t_2)\}X(t_2)\big] \qquad (7\text{-}39)\\ &= 0 + 2D\, t_2 .\end{aligned}$$

By symmetry, we can conclude that

$$R_X(t_1, t_2) = 2D\{\min(t_1,t_2)\}, \;\; t_1 \;\&\; t_2 \ge 0, \qquad (7\text{-}40)$$

for the Wiener process $X(t)$, $t \ge 0$, with $X(0) = 0$.

Correlation Time

Let $X(t)$ be a zero mean (*i.e.,* $E[X(t)] = 0$) W.S.S. random process. The ***correlation time*** of $X(t)$ is defined as

$$\tau_x \equiv \frac{1}{R_x(0)}\int_0^\infty \left|R_x(\tau)\right| d\tau . \qquad (7\text{-}41)$$

Intuitively, time τ_x gives some measure of the time interval over which "significant" correlation exists between two samples of process $X(t)$.

For example, consider the random telegraph signal described above. For this process the correlation time is

$$\tau_x \equiv \frac{1}{1}\int_0^{\infty} e^{-2\lambda\tau}\, d\tau = \frac{1}{2\lambda}. \tag{7-42}$$

In Chapter 8, we will relate correlation time to the spectral bandwidth (to be defined in Chapter 8) of a W.S.S. process.

Crosscorrelation Functions

Let X(t) and Y(t) denote real-valued random processes. The *crosscorrelation* of X and Y is defined as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y; t_1, t_2)\, dx\, dy. \tag{7-43}$$

The *crosscovariance* function is defined as

$$C_{XY}(t_1, t_2) = E[\{X(t_1) - \eta_X(t_1)\}\{Y(t_2) - \eta_Y(t_2)\}] = R_{XY}(t_1, t_2) - \eta_X(t_1)\eta_Y(t_2) \tag{7-44}$$

Let X(t) and Y(t) be WSS random processes. Then X(t) and Y(t) are said to be *jointly stationary in the wide sense* if $R_{XY}(t_1,t_2) = R_{XY}(\tau)$, $\tau = t_1 - t_2$. For jointly stationary in the wide sense processes the crosscorrelation is

$$R_{XY}(\tau) = E[X(t+\tau)Y(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x, y; \tau)\, dxdy \tag{7-45}$$

Warning: Some authors define $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$; in the literature, there is controversy in the definition of R_{XY} over which function is shifted. For R_{XY}, the order of the subscript is significant! In general, $R_{XY} \neq R_{YX}$.

For jointly stationary random processes X and Y, we show some elementary properties of the cross correlation function.

1. $R_{YX}(\tau) = R_{XY}(-\tau)$. To see this, note that

$$R_{YX}(\tau) = E[Y(t+\tau)X(t)] = E[Y(t-\tau+\tau)X(t-\tau)] = E[Y(t)X(t-\tau)] = R_{XY}(-\tau)\,. \tag{7-46}$$

2. $R_{YX}(\tau)$ does not necessarily have its maximum at $\tau = 0$; the maximum can occur anywhere. However, we can say that

$$2\left|R_{XY}(\tau)\right| \le R_X(0) + R_Y(0)\,. \tag{7-47}$$

To see this, note that

$$\begin{aligned} E[\{X(t+\tau) \pm Y(t)\}^2] &= E[X^2(t+\tau)] \;\pm\; 2E[X(t+\tau)Y(t)] \;+ E[Y^2(t)] \ge 0 \\ &= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \ge 0 \end{aligned}\,. \tag{7-48}$$

Hence, we have $2\left|R_{XY}(\tau)\right| \le R_X(0) + R_Y(0)$ as claimed.

Linear, Time-Invariant Systems: Expected Value of the Output

Consider a linear time invariant system with impulse response h(t). Given input X(t), the output Y(t) can be computed as

$$Y(t) = L\big[X(t)\big] \equiv \int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau \tag{7-49}$$

The notation L[•] denotes a linear operator (the convolution operator in this case). As given by (7-49), output Y(t) depends only on input X(t), initial conditions play no role here (assume that all initial conditions are zero).

Convolution and expectation are integral operators. In applications that employ these operations, it is assumed that we can interchange the order of convolution and expectation. Hence, we can write

$$\begin{aligned} E[Y(t)] &= E\big[L[X(t)]\big] \equiv E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,d\tau\right] \\ &= \int_{-\infty}^{\infty} E[X(\tau)]h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \eta_X(\tau)h(t-\tau)\,d\tau. \end{aligned} \tag{7-50}$$

More generally, in applications, it is assumed that we can interchange the operations of expectation and integration so that

$$E\left[\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_n}^{\beta_n} f(t_1,\cdots,t_n)dt_1\cdots dt_n\right] = \int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_n}^{\beta_n} E\big[f(t_1,\cdots,t_n)\big]dt_1\cdots dt_n\,, \tag{7-51}$$

for example (f is a random function involving variables $t_1, \ldots, t_n$).

As a special case, assume that input X(t) is wide-sense stationary with mean η_x. Equation (7-50) leads to

$$\eta_Y = E[Y(t)] = \eta_X\left[\int_{-\infty}^{\infty} h(t-\tau)\,d\tau\right] = \eta_X\left[\int_{-\infty}^{\infty} h(\tau)\,d\tau\right] = \eta_X H(0)\,, \tag{7-52}$$

where H(0) is the DC response (*i.e.*, DC gain) of the system.

Linear, Time-Invariant Systems: Input/Output Cross Correlation

Let $R_X(t_1,t_2)$ denote the autocorrelation of input random process X(t). We desire to find $R_{XY}(t_1,t_2) = E[x(t_1)y(t_2)]$, the crosscorrelation between input X(t) and output Y(t) of a linear, time-invariant system.

Theorem 7-1

The cross correlation between input X(t) and output Y(t) can be calculated as (both X and Y are assumed to be real-valued)

$$R_{XY}(t_1,t_2) = E\big[X(t_1)Y(t_2)\big] = L_2\big[R_X(t_1,t_2)\big] = \int_{-\infty}^{\infty} R_X(t_1,t_2-\alpha)\,h(\alpha)\,d\alpha \tag{7-53}$$

Notation: $L_2[\cdot]$ means operate on the t_2 variable (the second variable) and treat t_1 (the first variable) as a fixed parameter.

Proof: In (7-53), the convolution involves folding and shifting the "t_2 slot" so we write

$$Y(t_2) = \int_{-\infty}^{\infty} X(t_2-\alpha)h(\alpha)\,d\alpha \Longrightarrow X(t_1)Y(t_2) = \int_{-\infty}^{\infty} X(t_1)X(t_2-\alpha)h(\alpha)\,d\alpha\,, \tag{7-54}$$

a result that can be used to derive

$$\begin{aligned} R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} E[X(t_1)X(t_2-\alpha)]\,h(\alpha)\,d\alpha \\ &= \int_{-\infty}^{\infty} R_X(t_1,t_2-\alpha)\,h(\alpha)\,d\alpha \\ &= L_2\big[R_X(t_1,t_2)\big]. \end{aligned} \tag{7-55}$$

Special Case: X is WSS

Suppose that input process X(t) is WSS. Let $\tau = t_1 - t_2$ and write (7-55) as

$$\begin{aligned} R_{XY}(\tau) &= \int_{-\infty}^{\infty} R_X(\tau+\alpha)\,h(\alpha)\,d\alpha = \int_{-\infty}^{\infty} R_X(\tau-\alpha)\,h(-\alpha)\,d\alpha \\ &= R_X(\tau) * h(-\tau) \end{aligned} \tag{7-56}$$

Note that X(t) and Y(t) are jointly wide sense stationary.

Theorem 7-2

The autocorrelation $R_Y(\tau)$ can be obtained from the crosscorrelation $R_{XY}(\tau)$ by the formula

$$R_Y(t_1,t_2) = L_1\big[R_{XY}(t_1,t_2)\big] = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2)\, h(\alpha)\, d\alpha \tag{7-57}$$

Notation: $L_1[\cdot]$ means operate on the t_1 variable (the first variable) and treat t_2 (the second variable) as a fixed parameter.

Proof: In (7-57), the convolution involves folding and shifting the "t_1 slot" so we write

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1-\alpha)h(\alpha)\, d\alpha \Longrightarrow Y(t_1)Y(t_2) = \int_{-\infty}^{\infty} Y(t_2)X(t_1-\alpha)h(\alpha)\, d\alpha \tag{7-58}$$

Now, take the expected value of (7-58) to obtain

$$\begin{aligned} R_Y(t_1,t_2) &= E[Y(t_1)Y(t_2)] \\ &= \int_{-\infty}^{\infty} E[Y(t_2)X(t_1-\alpha)]\, h(\alpha)\, d\alpha = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2)\, h(\alpha)\, d\alpha \\ &= L_1\big[R_{XY}(t_1,t_2)\big], \end{aligned} \tag{7-59}$$

a result that completes the proof of our theorem.

The last two theorems can be combined into a single formula for finding $R_Y(t_1,t_2)$. Consider the formula

$$\begin{aligned} R_Y(t_1,t_2) &= \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2)\, h(\alpha)\, d\alpha \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_X(t_1-\alpha,t_2-\beta)\, h(\beta)\, d\beta\right] h(\alpha)\, d\alpha\ . \end{aligned} \tag{7-60}$$

This result leads to

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-\alpha, t_2-\beta)\, h(\alpha)h(\beta)\, d\alpha\, d\beta, \tag{7-61}$$

an important "double convolution" formula for R_Y in terms of R_X.

Special Case: X(t) is W.S.S.

Suppose that input process X(t) is WSS. Let $\tau = t_1 - t_2$ and write (7-61) as

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-t_2-[\alpha-\beta])\, h(\alpha)h(\beta)\, d\alpha\, d\beta \tag{7-62}$$

Define $\tau \equiv t_1 - t_2$; in (7-62) change the variables of integration to α and $\gamma \equiv \alpha - \beta$ and obtain

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\gamma)\, h(\alpha)h(\alpha-\gamma)\, d\gamma\, d\alpha \\ &= \int_{-\infty}^{\infty} R_X(\tau-\gamma)\left[\int_{-\infty}^{\infty} h(\alpha)h(-[\gamma-\alpha])\, d\alpha\right] d\gamma \end{aligned} . \tag{7-63}$$

This last formula can be expressed in a more convenient form. First, define

$$\Psi(\tau) \equiv \int_{-\infty}^{\infty} h(\alpha)h(-[\tau-\alpha])\, d\alpha = h(\tau) * h(-\tau). \tag{7-64}$$

Then, (7-63) can be expressed as

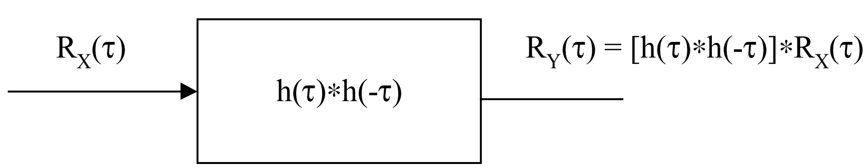

Fig. 7-7: Output autocorrelation in terms of input autocorrelation.

$$R_Y(\tau) = \int_{-\infty}^{\infty} R_X(\tau-\gamma)\Psi(\gamma)\,d\gamma = R_X(\tau)*\Psi(\tau)\,, \tag{7-65}$$

a result that is illustrated by Figure 7-7. Equation (7-65) is a convenient formula for computing $R_Y(\tau)$ when input X(t) is W.S.S. Note from (7-65) that WSS input X(t) produce WSS output Y(t). A similar statement can be made for stationary in the strict sense: strict-sense stationary input X(t) produces strict-sense stationary output Y(t).

Example 7-3

A zero-mean, stationary process X(t) with autocorrelation $R_x(\tau) = q\delta(\tau)$ (white noise) is applied to a linear system with impulse response $h(t) = e^{-ct}U(t)$, $c > 0$. Find $R_{xy}(\tau)$ and $R_y(\tau)$ for this system. Since X is WSS, we know that

$$\begin{aligned} R_{XY}(\tau) &= E[X(t+\tau)Y(t)] = R_X(\tau)*h(-\tau) = q\delta(\tau)*e^{c\tau}U(-\tau) \\ &= q\,e^{c\tau}U(-\tau). \end{aligned} \tag{7-66}$$

Note that X and Y are jointly WSS. That $R_{XY}(\tau) = 0$ for $\tau > 0$ should be intuitive since X(t) is a white noise process. Now, the autocorrelation of Y can be computed as

$$\begin{aligned} R_Y(\tau) &= E[Y(t+\tau)Y(t)] = R_X(\tau)*[h(\tau)*h(-\tau)] = [R_X(\tau)*h(-\tau)]*h(\tau) \\ &= R_{XY}(\tau)*h(\tau) = \{qe^{c\tau}U(-\tau)\}*\{e^{-c\tau}U(\tau)\} \\ &= \frac{q}{2c}e^{-c|\tau|}\,, \quad -\infty < \tau < \infty. \end{aligned} \tag{7-67}$$

Note that output Y(t) is not "white" noise; samples of Y(t) are correlated with each other. Basically, system h(t) filtered white-noise input X(t) to produce an output Y(t) that is correlated. As we will see in Chapter 8, input X(t) ***is modeled*** as having an infinite bandwidth; the system "bandlimited" its input to form an output Y(t) that has a finite bandwidth.

Example 7-4 (from Papoulis, 3rd Ed., pp. 311-312)

A zero-mean, stationary process X(t) with autocorrelation $R_x(\tau) = q\delta(\tau)$ (white noise) is applied **at t = 0** to a linear system with impulse response $h(t) = e^{-ct}U(t)$, $c > 0$. See Figure 7-8. Assume that the system is "at rest" initially (the initial conditions are zero so that Y(t) = 0, t < 0). Find R_{xy} and R_y for this system.

In a subtle way, this problem differs from the previous example. Since the input is applied at t = 0, the system "sees" a non-stationary input. Hence, we must analyze the general, nonstationary case. As shown below, processes X(t) and Y(t) **are not** jointly wide sense stationary, and Y(t) **is not** wide sense stationary. Also, it should be obvious that $R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = 0$ for $t_1 < 0$ or $t_2 < 0$ (note how this differs from Example 7-3).

$R_{xy}(t_1,t_2)$ equals the response of the system to $R_x(t_1-t_2) = q\delta(t_1-t_2)$, when t_1 is held fixed and t_2 is the independent variable (the input δ function occurs at $t_2 = t_1$). For $t_1 > 0$ and $t_2 > 0$, we can write

$$
\begin{aligned}
R_{xy}(t_1,t_2) &= L_2\left[R_X(t_1,t_2)\right] = \int_{-\infty}^{\infty} R_X(t_1,t_2-\alpha)\,h(\alpha)\,d\alpha \\
&= \int_{-\infty}^{\infty} q\,\delta(t_1-t_2+\alpha)\,h(\alpha)\,d\alpha = q\,h(-[t_1-t_2]) \\
&= q\exp[-c(t_2-t_1)]\,U(t_2-t_1), \qquad t_1>0,\ t_2>0, \\
&= 0, \qquad t_1<0 \text{ or } t_2<0,
\end{aligned}
\tag{7-68}
$$

a result that is illustrated by Figure 7-9. For $t_2 < t_1$, output $Y(t_2)$ is uncorrelated with input $X(t_1)$, as expected (this should be intuitive). Also, for $(t_2 - t_1) > 5/c$ we can assume that $X(t_1)$ and $Y(t_2)$

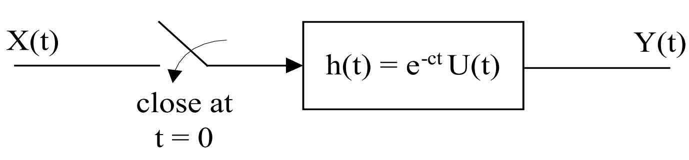

Figure 7-8: System with random input applied at t = 0.

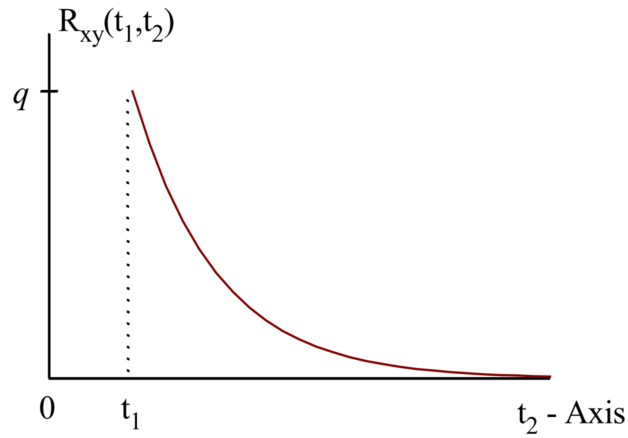

Figure 7-9: Plot of **(7-68)**; the crosscorrelation between input X and output Y.

are uncorrelated. Finally, note that X(t) and Y(t) are not jointly wide sense stationary since R_{XY} depends on absolute t_1 and t_2 (and not only the difference $\tau = t_1 - t_2$).

Now, find the autocorrelation of the output; there are two cases. The first case is $t_2 > t_1 > 0$ for which we can write

$$
\begin{aligned}
R_Y(t_1,t_2) &= E[Y(t_1)Y(t_2)] \\
&= \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2)\, h(\alpha)\, d\alpha \\
&= \int_0^{t_1} q\, e^{-c(t_2-[t_1-\alpha])} e^{-c\alpha} U(t_2-[t_1-\alpha]) d\alpha \\
&= \frac{q}{2c}(1-e^{-2ct_1})e^{-c(t_2-t_1)}, \quad t_2 > t_1 > 0,
\end{aligned}
\tag{7-69}
$$

Note that the requirements 1) $h(\alpha) = 0$, $\alpha < 0$, and 2) $t_1 - \alpha > 0$ were used to write (7-69). The second case is $t_1 > t_2 > 0$ for which we can write

$$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2)\,h(\alpha)\,d\alpha$$

$$\begin{aligned}
&= \int_0^{t_1} q\,e^{-c(t_2-[t_1-\alpha])}e^{-c\alpha}U(t_2-[t_1-\alpha])d\alpha \\
&= \int_{t_1-t_2}^{t_1} q\,e^{-c(t_2-[t_1-\alpha])}e^{-c\alpha}d\alpha \qquad (7\text{-}70) \\
&= \frac{q}{2c}(1-e^{-2ct_2})e^{-c(t_1-t_2)}, \quad t_1 > t_2 > 0.
\end{aligned}$$

Note that output Y(t) *is not stationary*. The reason for this is simple (and intuitive). Input X(t) is applied at t = 0, and the system is "at rest" before this time (Y(t) = 0, t < 0). For a few time constants, this fact is "remembered" by the system (the system "has memory"). For t_1 and t_2 larger than 5 time constants (t_1, t_2 > 5/(2c)), "steady state" can be assumed, and the output autocorrelation can be approximated as

$$R_y(t_1,t_2) \approx \frac{q}{2c}e^{-c|t_2-t_1|}. \qquad (7\text{-}71)$$

Output y(t) is "approximately stationary" for t_1, t_2 > 5/(2c_1).

Example 7-5: Let X(t) be a real-valued, WSS process with autocorrelation R(τ). For any fixed T > 0, define the random variable

$$S_T \equiv \int_{-T}^{T} X(t)dt\,. \qquad (7\text{-}72)$$

Express the second moment $E[S_T^2]$ as a single integral involving R(τ). First, note that

$$S_T^2 \equiv \int_{-T}^{T} X(t_1)dt_1 \int_{-T}^{T} X(t_2)dt_2 = \int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)dt_1dt_2\,, \qquad (7\text{-}73)$$

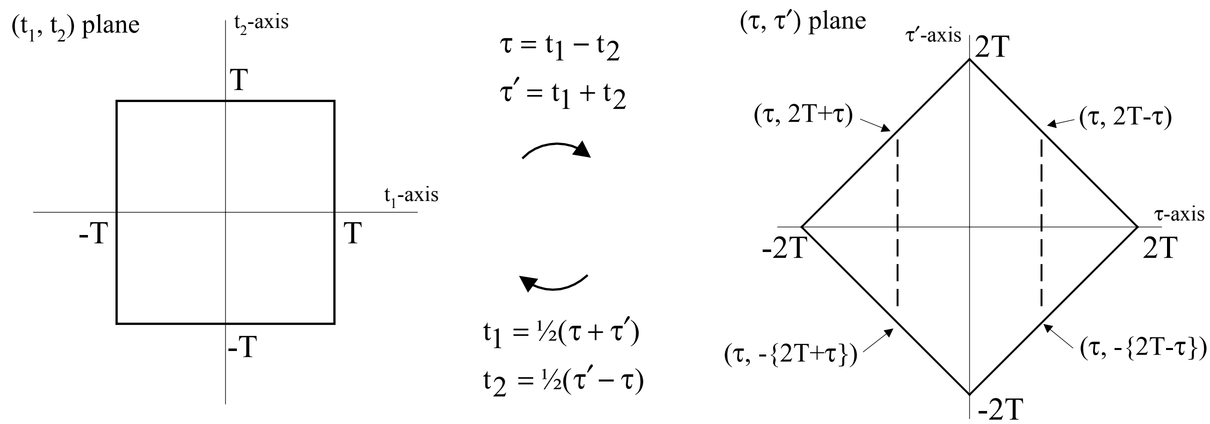

Fig. 7-10: Geometry used to change variables from (t_1, t_2) to (τ, τ') in the double integral that appears in Example 7-5.

a result that leads to

$$E\left[S_T^2\right] = \int_{-T}^{T}\int_{-T}^{T} E\left[X(t_1)X(t_2)\right]dt_1dt_2 = \int_{-T}^{T}\int_{-T}^{T} R(t_1 - t_2)\, dt_1dt_2\,. \tag{7-74}$$

The integrand in (7-74) depends only on one quantity, namely the difference $t_1 - t_2$. Therefore, Equation (7-74) should be expressible in terms of a **single integral** in the variable $\tau \equiv t_1 - t_2$. To see this, use $\tau \equiv t_1 - t_2$ and $\tau' = t_1 + t_2$ and map the (t_1, t_2) plane to the (τ, τ') plane (this relationship has an inverse), as shown by Fig. 7-10. As discussed in Appendix 4A, the integral (7-74) can be expressed as

$$\int_{-T}^{T}\int_{-T}^{T} R(t_1 - t_2)\,dt_1dt_2 = \iint\limits_{\mathcal{R}_2} R(\tau)\left|\frac{\partial(t_1, t_2)}{\partial(\tau, \tau')}\right| d\tau\, d\tau', \tag{7-75}$$

where $\mathcal{R}_2$ is the "rotated square" region in the (τ, τ') plane shown on the right-hand side of Fig. 7-10. For use in (7-75), the Jacobian of the transformation is

$$\frac{\partial(t_1,t_2)}{\partial(\tau,\tau')} = \begin{vmatrix} ½ & ½ \\ -½ & ½ \end{vmatrix} = ½ \tag{7-76}$$

In the (τ,τ')-plane, as τ goes from $-2T$ to 0, the quantity τ' traverses from $-2T-\tau$ to $2T+\tau$, as can be seen from examination of Fig. 7-10. Also, as τ goes from 0 to 2T, the quantity τ' traverses from $-2T+\tau$ to $2T-\tau$. Hence, we have

$$E\left[S_T^2\right] = \int_{-T}^{T}\int_{-T}^{T} R(t_1-t_2)\,dt_1 dt_2 = \int_{-2T}^{0}\int_{-(2T+\tau)}^{2T+\tau} ½R(\tau)\,d\tau' d\tau + \int_{0}^{2T}\int_{-(2T-\tau)}^{2T-\tau} ½R(\tau)\,d\tau' d\tau$$

$$= \int_{-2T}^{0}(2T+\tau)R(\tau)\,d\tau + \int_{0}^{2T}(2T-\tau)R(\tau)\,d\tau \tag{7-77}$$

$$= \int_{-2T}^{2T}(2T-|\tau|)R(\tau)\,d\tau$$

a single integral that can be evaluate given autocorrelation function $R(\tau)$.