Appendix 9A: Hilbert Transforms

Consider the filter H(ω), described by Figure 9A-1, that has a unity magnitude response for all frequencies. Also, the phase response is -π/2 for positive frequencies and π/2 for negative frequencies. The transfer function of this filter is

$$H(j\omega) = -j\,\mathrm{sgn}(\omega)\,. \tag{9A1}$$

An engineer might think of filter H(ω) as a wide-band phase shift network.

The impulse response of the filter is

$$h(t) = \mathcal{F}^{-1}[H] = -j\mathcal{F}^{-1}[\mathrm{sgn}(\omega)] = -j\left(\frac{j}{\pi t}\right) = \frac{1}{\pi t}\,. \tag{9A2}$$

When driven by an arbitrary signal x(t), the filter produces the output

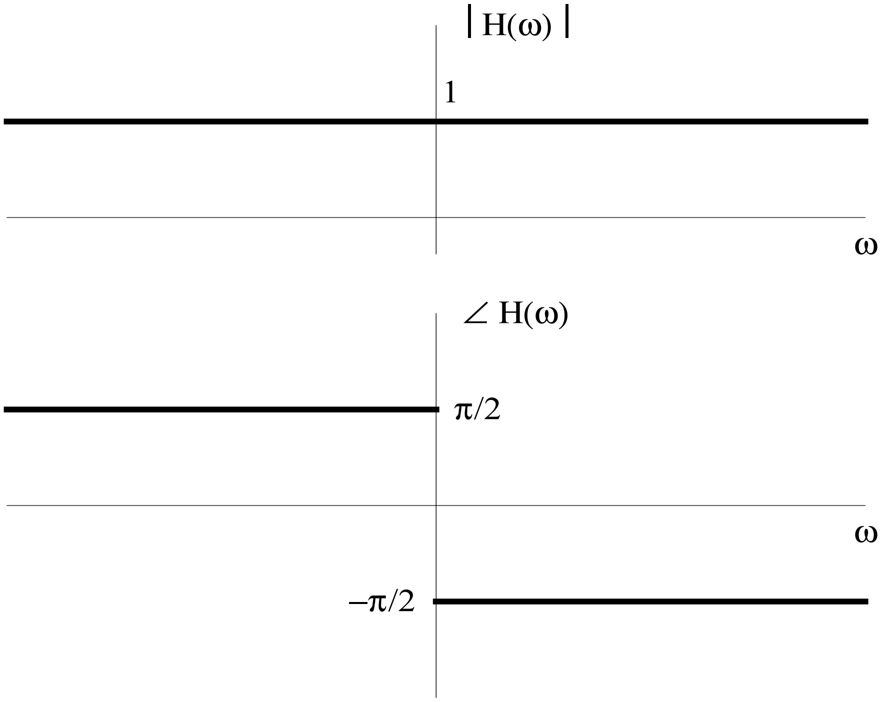

Figure 9A-1: Magnitude and phase of Hilbert transform operator.

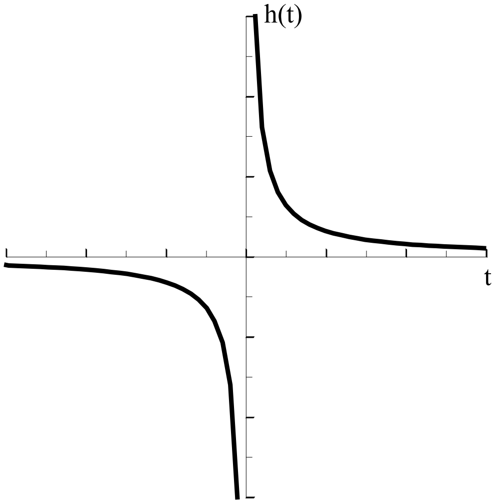

Figure 9A-2: Impulse response h(t) of Hilbert transform operator

$$\hat{x}(t) = x * h = \int_{-\infty}^{\infty} \frac{x(u)}{\pi(t-u)}\,du\,. \tag{9A3}$$

The function $\hat{x}(t)$ is the *Hilbert Transform* of x(t). Note that

$$\mathcal{F}[\hat{x}] = H(\omega)X(\omega) = -j\,\mathrm{sgn}(\omega)X(\omega)\,, \tag{9A4}$$

so that

$$\hat{x}(t) = \mathcal{F}^{-1}\big[-j\,\mathrm{sgn}(\omega)X(\omega)\big]. \tag{9A5}$$

In some cases, this formula allows use of a Fourier transform table to compute the Hilbert transform.

EXAMPLES

1. Consider $x(t) = \cos(\omega_0 t)$ with transform $X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$. We have

$$-j\operatorname{sgn}(\omega)X(\omega) = j\pi\big[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\big],\ \omega_0 > 0$$

$$= j\pi\big[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\big],\ \omega_0 < 0$$

so that $-j\operatorname{sgn}(\omega)X(\omega) = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]\operatorname{sgn}(\omega_0)$. Hence, we can write

$$\hat{x}(t) = \widehat{\cos(\omega_0 t)} = \mathcal{F}^{-1}\big[-j\operatorname{sgn}(\omega)X(\omega)\big] = \operatorname{sgn}(\omega_0)\sin(\omega_0 t)\,. \tag{9A6}$$

2. In a similar manner, we can write

$$\widehat{\sin(\omega_0 t)} = -\operatorname{sgn}(\omega_0)\cos(\omega_0 t) \tag{9A7}$$

3. Combine (9A6) and (9A7) to obtain

$$\widehat{\exp\{j\omega_0 t\}} = \widehat{\cos\omega_0 t + j\sin\omega_0 t} = \operatorname{sgn}(\omega_0)[\sin\omega_0 t - j\cos\omega_0 t] = -j\operatorname{sgn}(\omega_0)\exp\{j\omega_0 t\} \tag{9A8}$$

Properties of Hilbert Transforms

1. The energy (or power) in x(t) and $\hat{x}(t)$ are equal. This claim follows from

$$\left|\mathcal{F}[\hat{x}]\right|^2 = \left|-j\operatorname{sgn}(\omega)X(\omega)\right|^2 = \left|X(\omega)\right|^2 = \left|\mathcal{F}[x]\right|^2. \tag{9A9}$$

Since the energy (or power) density spectrum at the input and output of the filter are the same, the two energies (or powers) are equal.

2. $\hat{\hat{x}}(t) = -x(t)$. This claim follows from

$$\begin{aligned}\hat{\hat{x}}(t) &= \mathcal{F}^{-1}\left[-j\operatorname{sgn}(\omega)\mathcal{F}[\hat{x}(t)]\right] = \mathcal{F}^{-1}\big[-j\operatorname{sgn}(\omega)\left[-j\operatorname{sgn}(\omega)X(j\omega)\right]\big]\\ &= \mathcal{F}^{-1}\left[-X(j\omega)\right] = -x(t).\end{aligned} \tag{9A10}$$

3. x(t) and $\hat{x}(t)$ are orthogonal. For energy signals, we have

$$\lim_{T\to\infty}\int_{-T}^{T} x(t)\hat{x}(t)dt = 0 . \tag{9A11}$$

For power signals, we have

$$\langle x(t)\hat{x}(t)\rangle \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\hat{x}(t)dt = 0 . \tag{9A12}$$

This claim follows from (proof given for energy signals; case of power signals is similar)

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} -j\,\mathrm{sgn}(\omega)X(j\omega)e^{j\omega t}d\omega\right]dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} -j\,\mathrm{sgn}(\omega)X(j\omega)\left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right]d\omega \\ &= \frac{-j}{2\pi}\int_{-\infty}^{\infty} \mathrm{sgn}(\omega)\left|X(j\omega)\right|^2 d\omega \\ &= 0 ,\end{aligned} \tag{9A13}$$

since integrand sgn(ω) $|X(j\omega)|^2$ is an odd function which is integrated over symmetric limits.

4. If c(t) and m(t) are signals with non-overlapping spectra, where m(t) is low pass and c(t) is high pass, then

$$\widehat{m(t)c(t)} = m(t)\hat{c}(t) \tag{9A14}$$

To develop this important result, denote **M**(ω) = $\mathcal{F}$[m(t)] and **C**(ω) = $\mathcal{F}$[c(t)] as the Fourier transform of m and c, respectively. The fact that the signals have no over-lapping spectrum implies that there exists a W for which

$$\begin{aligned} \mathbf{M}(\omega) &= \mathcal{F}[\mathrm{m}(t)] = 0, \ |\omega| > W \\ \mathbf{C}(\omega) &= \mathcal{F}[\mathrm{c}(t)] = 0, \ |\omega| < W \end{aligned} \tag{9A15}$$

since m(t) is low pass and c(t) is high pass. Use (9A8) and note that

$$\begin{aligned} \widehat{\mathrm{m}(t)\mathrm{c}(t)} &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{M}(\omega_1)\mathbf{C}(\omega_2)\,\widehat{\exp[j(\omega_1+\omega_2)]}\,d\omega_1 d\omega_2 \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{M}(\omega_1)\mathbf{C}(\omega_2)[-j\,\mathrm{sgn}(\omega_1+\omega_2)]\exp[j(\omega_1+\omega_2)]\,d\omega_1 d\omega_2 \end{aligned} \tag{9A16}$$

Once the quantity sgn(ω_1 + ω_2) in the integrand of (9A16) is simplified, we will obtain the desired result. To simplify (9A16), note that non-overlapping spectra and (9A15) imply

$$\begin{aligned} \mathbf{M}(\omega_1) &= 0, \ |\omega_1| > W \\ \mathbf{C}(\omega_2) &= 0, \ |\omega_2| < W \end{aligned} . \tag{9A17}$$

Hence, the integrand of (9A16) is zero for all (ω_1, ω_2) in the cross-hatched region on Fig 9A-3. More importantly, on the shaded region of the (ω_1, ω_2) plane, the integrand is non-zero, and we can write

$$\mathrm{sgn}(\omega_1+\omega_2) = \mathrm{sgn}(\omega_2) \tag{9A18}$$

for all (ω_1, ω_2) in the shaded (but not the cross-hatched!) region illustrated on Figure 9A-3. Finally, use of simplification (9A18) in Equation (9A16) yields the desired result

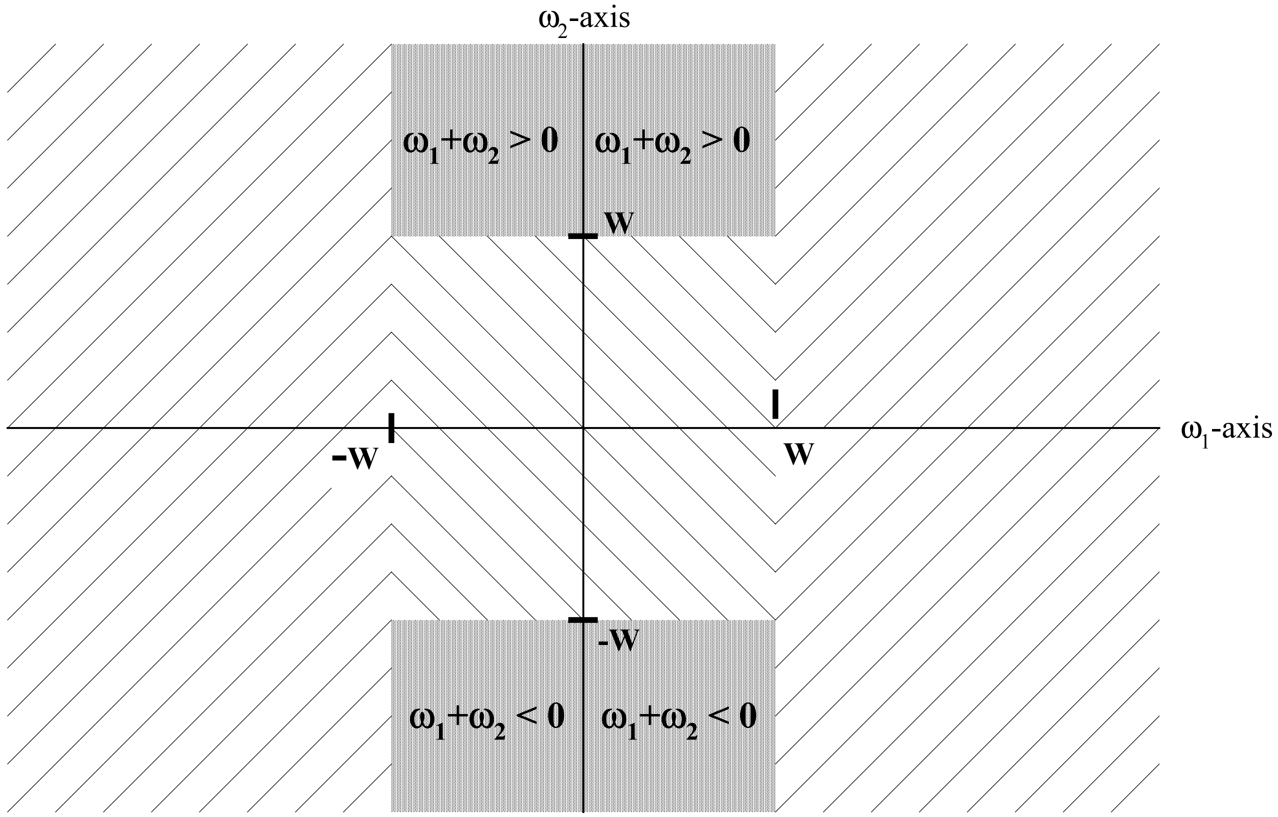

Figure 9A-3: Integrand of **(9A16)** is zero in the cross-hatched region. In the upper-half plane shaded region, we have $U(\omega_1 + \omega_2) = 1$. In the lower-half plane shaded region, we have $U(\omega_1 + \omega_2) = -1$.

$$\begin{aligned}\widehat{m(t)c(t)} &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{M}(\omega_1)\exp[j\omega_1 t]d\omega_1\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C}(\omega_2)[-j\mathrm{sgn}(\omega_2)]\exp[j\omega_2]\,d\omega_2\right]\\ &= m(t)\hat{c}(t)\end{aligned}$$

which is (9A14).

5) Since the impulse response h(t) does not vanish for t < 0, *the Hilbert transform is a non-causal linear operator*.

6. If x(t) is an even (alternatively, odd) function then $\hat{x}(t)$ is an odd (alternatively, even) function. This claim follows easily from Fourier transform theory. If x(t) is even, then $X(j\omega) = \mathcal{F}[x(t)]$ is real-valued. As a result, $-j\mathrm{sgn}(\omega)X(j\omega)$ is purely imaginary. But this means that $\hat{x}(t) = \mathcal{F}\left[-j\mathrm{sgn}(\omega)X(j\omega)\right]$ will be odd.