Appendix 4.2: Hermitian Matrices

A square n×n matrix B is said to be *Hermitian* if **B*** = B. Here, the * denotes complex-conjugate transpose (some authors use an "H" as a subscript to denote complex-conjugate transpose, and they would write $B^H = B$). We need two important attributes of Hermitian matrices. First, their eigenvalues are always real-valued. Secondly, they are *unitary similar* to a diagonal matrix containing the eigenvalues. That is, there exists an n×n *unitary* matrix U (*i.e.*, U* = U^{-1} or U*U = I) such that U*BU is a diagonal matrix with the eigenvalues of B on its diagonal.

Theorem 4.2.1

A Hermitian matrix has real-valued eigenvalues.

Proof: Let λ and $\vec{X}$ be an eigenvalue and eigenvector, respectively, of B. On the left, multiply $B\vec{X} = \lambda\vec{X}$ by $\vec{X}^*$ to obtain

$$\vec{X}^* B\vec{X} = \lambda\vec{X}^*\vec{X} \tag{4.2-1}$$

Now, $\vec{X}^*\vec{X}$ is *always* real-valued . Also, $\vec{X}^*B\vec{X}$ has to be real-valued since $\{\vec{X}^*B\vec{X}\}^* = \vec{X}^*B^*\vec{X} = \vec{X}^*B\vec{X}$. By inspection of the last equation, we conclude that λ must be real-valued.♥

Let B be a Hermitian matrix (B* = B). As we know, it has real eigenvalues. Also, it is unitary similar to a diagonal matrix containing eigenvalues on the diagonal.

Theorem 4.2.2

Let n×n matrix B be Hermitian. Then there exists an n×n unitary matrix U (*i.e.*, U*U = I) such that $U^{-1}BU$ = U*BU is a diagonal matrix with the eigenvalues of B on its diagonal. Furthermore, the eigenvectors of B are the columns of U.

Proof: By the previous theorem, there exists an n×n unitary matrix U such that U*BU is upper-triangular with the eigenvalues on the diagonal (i.e., U*BU is in Schur Form). However, (U*BU)* = U*B*U = U*BU, so U*BU is Hermitian. Note that an upper-triangular, Hermitian matrix must necessarily be diagonal. Hence $D \equiv U^*BU$ is diagonal with eigenvalues on its diagonal. From BU = DU, it is clear that the columns of U are the eigenvectors of B.♥

According to Theorem 4.2-1, matrix D has only real-valued eigenvalues. According to Theorem 4.2-2, matrix D looks like

$$D = U^*BU = \begin{bmatrix} \lambda_1 & & & & \bigcirc & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_r & & \\ & \bigcirc & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}, \tag{4.2-2}$$

where λ_k, $1 \le k \le r \le n$, are the non-zero eigenvalues. *If* $\lambda_k > 0$, $1 \le k \le n$, then Hermitian B is said to be *positive definite*, and this is denoted by writing B > 0. In this case, it is easily shown by using the diagonal decomposition that $\vec{X}^*B\vec{X} > 0$ for all $\vec{X} \ne \vec{0}$. *If* $\lambda_k \ge 0$, $1 \le k \le n$, then Hermitian B is said to be *nonnegative definite*, and this is denoted by writing $B \ge 0$. In this case, it is easily shown by using the diagonal decomposition that $\vec{X}^*B\vec{X} \ge 0$ for all $\vec{X}$. Clearly, the positive definite attribute is "stronger" than the nonnegative definite attribute. A positive definite matrix is also nonnegative definite, but the converse is not true.

MatLab Example: Schur Decomposition of a Hermitian Matrix

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% Enter a Hermitian matrix B
> B = [1 2+3j 5j;2-3j 2 1+7j;-5j 1-7j 3]
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$$B = \begin{bmatrix} 1 & (2+3j) & +5j \\ (2-3j) & 2 & (1+7j) \\ -5j & (1-7j) & 3 \end{bmatrix}$$

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% The eigenvalues of B are real-valued (as is true for the eigenvalues of any Hermitian matrix!).
> eig(B)
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12.4327, -0.5018, -5.9309

> [U,T]=schur(B)

$$U = \begin{bmatrix} 0.4514 & (-0.4251 - 0.6376j) & (-0.4489 - 0.0864j) \\ (0.5023 - 0.3105j) & (0.2035 + 0.4896j) & (-0.2789 - 0.5407j) \\ (-0.1772 - 0.6450j) & (-0.1264 - 0.3401j) & (0.5034 - 0.4093j) \end{bmatrix}$$

$$T = \begin{bmatrix} 12.4327 & 0 & 0 \\ 0 & -0.5018 & 0 \\ 0 & 0 & -5.9309 \end{bmatrix}$$

% It works! The Schur Form For Hermitian B is a diagonal matrix with the eigenvalues on the diagonal. ♥

Analogy Between Hermitian Matrices and Real Numbers

An analogy between Hermitian matrices and real numbers can be made. Each positive (alternatively, nonnegative) real number has a positive (alternatively, nonnegative) square root. A similar statement can be made for Hermitian matrices.

Theorem 4.2.3

An n×n Hermitian matrix H is positive (alternatively, nonnegative) definite if, and only if, there exists a positive (alternatively, nonnegative) definite Hermitian matrix H_0 such that $H_0^2 = H$. Matrix H_0 is called the *square root* of H.

Proof: (We prove the positive definite case; the nonnegative definite case is similar.)

Suppose that H is Hermitian and positive definite. Then it has eigenvalues λ_i, $1 \le i \le n$, that are real and positive. Furthermore, there exists a unitary matrix U such that $U^*HU = D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, a diagonal matrix with the eigenvalues of H on its diagonal. The square root of D is

$$D_0 = \sqrt{D} = \text{diag}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}\}, \tag{4.2-3}$$

a diagonal matrix with eigenvalues $\sqrt{\lambda_k}$, $1 \le k \le n$, on its diagonal. Note that matrix H can be written as

$$H = UD_0D_0U^* = \left(UD_0U^*\right)\left(UD_0U^*\right) = H_0^2, \tag{4.2-4}$$

where $H_0 = UD_0U^*$. We write $H_0 = \sqrt{H}$, and call H_0 the *square root* of Hermitian H. Note that the eigenvalues of H_0 are $\sqrt{\lambda_k}$, $1 \le k \le n$, all positive. Hence H_0 is a positive definite Hermitian matrix.

Conversely, suppose that $H = H_0^2$, where H_0 is a positive definite Hermitian matrix. Clearly, H is Hermitian; we show that H is positive definite. Let $\vec{X}$ and λ be an eigenvector and eigenvalue pair for H_0; note that λ is positive since H_0 is positive definite. Then we have

$$H\vec{X} = H_0^2\vec{X} = H_0\left(\lambda\vec{X}\right) = \lambda^2\vec{X}, \tag{4.2-5}$$

so that $\vec{X}$ and λ^2 is an eigenvector and eigenvalue pair for H. Hermitian matrix H has positive eigenvalues. Hence, it is positive definite.♥