Chapter 10

Function of a Matrix

Let f(z) be a complex-valued function of a complex variable z. Let A be an n×n complex-valued matrix. In this chapter, we give a definition for the n×n matrix f(A). Also, we show how f(A) can be computed. Our approach relies heavily on the Jordan canonical form of A, an important topic in Chapter 9. We do not strive for maximum generality in our exposition; instead, we want to get the job done as simply as possible, and we want to be able to compute f(A) with a minimum of fuss. In the literature, a number of equivalent approaches have been described for defining and computing a function of a matrix. The concept of a matrix function has many applications, especially in control theory and, more generally, differential equations (where exp(At) and ln(A) play prominent roles).

Function of an n×n Matrix

Let f(z) be a function of the complex variable z. We require that f(z) be analytic in the disk $|z| < R$. Basically (and for our purposes), this is equivalent to saying that f(z) can be represented as a convergent power series (Taylor's Series)

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \tag{10-1}$$

for $|z| < R$, where the c_k, $k \geq 0$, are complex-valued constants.

Formal substitution of n×n matrix A into the series (10-1) yields the symbolic result

$$f(A) = c_0 + c_1 A + c_2 A^2 + \cdots . \tag{10-2}$$

We say that this matrix series is convergent (to an n×n matrix f(A)) if *all* n^2 scalar series that make up f(A) are convergent. Now recall Theorem 4-1 which says, in part, that every element of a matrix has an absolute value that is bounded above by the 2-norm of the matrix. Hence, each element in f(A) is a series that is bounded in magnitude by the norm $\|f(A)\|_2$. But the norm of a sum is less than, or equal to, the sum of the norms. This leads to the conclusion that (10-2) converges if

$$\begin{aligned}\|f(A)\|_2 &= \| c_0 + c_1 A + c_2 A^2 + \cdots \|_2 \\ &\le |c_0| \|I\|_2 + |c_1| \|A\|_2 + |c_2| \|A^2\|_2 + |c_3| \|A^3\|_2 + \cdots \\ &\le |c_0| \|I\|_2 + |c_1| \|A\|_2 + |c_2| \|A\|_2^2 + |c_3| \|A\|_2^3 + \cdots\end{aligned} \tag{10-3}$$

converges. Now, the last series on the right of (10-3) is an "ordinary" power series; it converges for all n×n matrices A with $\|A\|_2 < R$. Hence, we have argued that f(A) can be represented by a series of matrices if $\|A\|_2$ is in the region of convergence of the scalar series (10-1).

At the end of this chapter, we will get a little more sophisticated. We will argue that the series for f(A) converges (*i.e.* f(A) exists) if all eigenvalues of A lie in the region of convergence of (10-1). That is, f(A) converges if $|\lambda_k| < R$, where λ_k, $1 \le k \le n$, are the eigenvalues of A. In addition, the series (10-1) diverges if one, or more, eigenvalues of A lie outside the disk $|z| < R$.

Our method of defining f(A) requires that function f(z) be analytic (so that it has a Taylor's series expansion) in some disk centered at the origin; here, we limit ourselves to working with "nice" functions This excludes from our analysis a number of interesting functions like $f(z) = z^{1/n}$, $n > 1$, and f(z) = ln(z), both of which have branch points at z = 0. While we do not cover it here, a matrix-function theory for these "more complicated" functions is available.

Sometimes, f(A) is "exactly what you think it should be". Roughly speaking, if the scalar function f(z) has a Taylor's series that converges in the disk $|z| < R$ containing the eigenvalues of A, then f(A) can be calculated by "substituting" matrix A for variable z in the formula for f(z). For example, if f(z) = (1+z)/(1-z), then $f(A) = (I + A)(I - A)^{-1}$, at least for matrices that have their eigenvalues inside a unit disk centered at the origin (actually, as can be shown by analytic continuation, this example is valid for all matrices A that do not have a unit eigenvalue). While this "direct substitution" approach works well with rational (and other simple) functions, it does not help compute transcendental functions like sin(A) and cos(A).

We are familiar with many elementary functions that have Taylor's series expansions that are convergent on the *entire* complex plane. For example, for any n×n matrix A, we can write

$$e^{A} = I + \sum_{k=1}^{\infty} A^{k} / k!$$

$$\cos(A) = I + \sum_{k=1}^{\infty} (-1)^{k} A^{2k} / (2k)! \tag{10-4}$$

$$\sin(A) = I + \sum_{k=1}^{\infty} (-1)^{k-1} A^{2k-1} / (2k-1)!,$$

to cite just a few. Also, as can be verified by the basic definition given above, many identities in the variable z remain valid for a matrix variable. For example, the common Euler's identity

$$e^{jA} = \cos(A) + j\sin(A) \tag{10-5}$$

is valid for any n×n matrix A.

Example

The following MatLab example illustrates a Taylor's series approximation to exp(A) where

$$A = \begin{bmatrix} .5 & 1 \\ 0 & .6 \end{bmatrix}.$$

The MatLab program listed below computes a ten-term approximation and compares the result with `expm(A)`, an accurate routine internal to MatLab.

```
%  Enter the matix A
> A = [.5 1;0 .6]
```

```
A = [0.5000   1.0000
       0      0.6000]
% Set Up Working Matix B = A
> B = A;
% Set Matrix f to the Identity Matrix
> f = [1 0; 0 1];
% Sum Ten Terms of the Taylor's Series
> for i = 1:10
> f = f + B;
> B = A*B/(i+1);
> end f
% Print-Out the Ten-Term Approximation to EXP[A]
f = [1.6487 1.7340
       0    1.8221]
% Use MatLab's Exponential Matix Function to Calculate EXP(A)
> expm(A)
ans = [1.6487 1.7340
         0    1.8221]
% Our Ten-Term Taylor's Series Approximation is
% Accurate to at Least 5 Decimal Digits!!!!
```

MatLab has several methods for computing exp[A]. For example, MatLab's `expm2(A)` function uses a Taylor's series to compute the exponential. The Taylor's series representation is good for introducing the concept of a matrix function. Also, many elementary analytical results come from the Taylor's expansion of f(A). However, direct implementation of the Taylor's series is a slow and inaccurate way for computing f(A).

Theorem 10-1

Let P be an n×n nonsingular matrix, and define $A' = P^{-1}AP \Leftrightarrow A = PA'P^{-1}$. Then, function f(A) exists if, and only if, f(A′) exists. Furthermore, we can write

$$f(A) = f(PA'P^{-1}) = Pf(A')P^{-1}. \qquad (10\text{-}6)$$

Proof This result is obvious once the Taylor's series of $f(PA'P^{-1})$ is written down.

Theorem 10-2

Let A be a block diagonal matrix

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix}, \qquad (10\text{-}7)$$

where A_k is an $n_k \times n_k$ square matrix (a block can be any size, but it must be square). Then we have

$$f(A) = \begin{bmatrix} f(A_1) & & & \\ & f(A_2) & & \\ & & \ddots & \\ & & & f(A_p) \end{bmatrix}. \qquad (10\text{-}8)$$

Proof: Obvious from examining the Taylor's Series of f(A).

Let A' be the Jordan canonical form of $n \times n$ matrix A. We can apply Theorems 10-1 and 10-2 to $A = P\,A'P^{-1}$ to obtain

$$f(A) = P\begin{bmatrix} f(J_1) & & & \\ & f(J_2) & & \\ & & \ddots & \\ & & & f(J_p) \end{bmatrix}P^{-1}, \qquad (10\text{-}9)$$

where J_k, $1 \le k \le p$, are the Jordan blocks of the Jordan form for A. Hence, to compute f(A), all we need to know is the Jordan form of A (*i.e.*, the blocks J_k, $1 \le k \le p$) and how to compute $f(J_k)$, where J_k is a Jordan block.

<u>Computing f(J), the Function of a Jordan Block</u>

As suggested by (10-9), function f(A) can be computed once we know how to compute a function of a Jordan block. Since a function of a block can be expressed as an infinite series of powers of a block (think of Taylor's series), we really need to know a simple formula for integer powers of a block (*i.e.*, we need to know a simple representation for J^p, the p^{th} power of block J).

But first, consider computing H^p, where $n_1 \times n_1$ matrix H has the form

$$H = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix}, \tag{10-10}$$

that is, $n_1 \times n_1$ matrix H has 1's on its **first** *superdiagonal* and zeroes everywhere else. To see the general trend, let's compute a few integer powers H. When $n_1 = 4$, we can compute easily

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H^p = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ p \ge 4 . \tag{10-11}$$

Notice that the all-one superdiagonal is on the p^{th} superdiagonal of H^p. For the 4×4 example illustrated by (10-11), the result becomes zero for $p \ge 4$. The general case can be inferred from this example. Consider raising $n_1 \times n_1$ matrix H (given by (10-10)) to an integer power p. For $p < n_1$, the result would have all 1's on the p^{th} superdiagonal and zero elsewhere. For $p \ge n_1$, the result is the $n_1 \times n_1$ all zero matrix.

For an integer $p > 0$, we can compute J^p, where J is an $n_1 \times n_1$ Jordan block associated with an eigenvalue λ. First, note that J can be written as

$$J = \lambda I + H, \tag{10-12}$$

where I is an $n_1 \times n_1$ identity matrix, and H is given by (10-10). Apply the Binomial expansion

$$(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^{p-k} y^k = x^p + p x^{p-1} y + \tfrac{1}{2!} p(p-1) x^{p-2} y^2 + \cdots + y^p \tag{10-13}$$

to (10-12) and write

$$J^p = (\lambda I + H)^p = \sum_{k=0}^{p} \binom{p}{k} \lambda^{p-k} H^k = \lambda^p I + p\lambda^{p-1} H + \tfrac{1}{2!} p(p-1)\lambda^{p-2} H^2 + \cdots + H^p . \tag{10-14}$$

On the far right-hand side of this expansion, the 1st term is the diagonal, the 2nd term is the first superdiagonal, the 3rd term is the second superdiagonal, and so forth. If $p < n_1$, the last term is the pth superdiagonal; on the other hand, if $p \ge n_1$, the last term is zero. Finally, note that (10-14) can be written as

$$J^p = \begin{bmatrix} \lambda^p & p\lambda^{p-1} & \tfrac{1}{2!} p(p-1)\lambda^{p-2} & \cdots & \tfrac{1}{(n_1-1)!} p(p-1)\cdots(p-[n_1-2])\lambda^{p-[n_1-1]} \\ 0 & \lambda^p & p\lambda^{p-1} & \cdots & \tfrac{1}{(n_1-2)!} p(p-1)\cdots(p-[n_1-3])\lambda^{p-[n_1-2]} \\ 0 & 0 & \lambda^p & \cdots & \tfrac{1}{(n_1-3)!} p(p-1)\cdots(p-[n_1-4])\lambda^{p-[n_1-3]} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p\lambda^{p-1} \\ 0 & 0 & 0 & \cdots & \lambda^p \end{bmatrix} . \tag{10-15}$$

Note that λ^p is on the diagonal, $p\lambda^p$ on the 1st superdiagonal, $\tfrac{1}{2!} p(p-1)\lambda^{p-2}$ on the second superdiagonal, $\frac{1}{3!}p(p-1)(p-2)\lambda^{p-3}$ on the third superdiagonal, and so on until the series terminates or you run out of superdiagonals to put terms on.

The matrix f(J) can be computed easily with the aid of (10-15). Expand f(J) in a Taylor's series to obtain

$$f(J) = c_0 + c_1 J + c_2 J^2 + \cdots \tag{10-16}$$

From (10-15) and (10-16) we can see that

$$f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & f''(\lambda)/2! & \cdots & f^{(n_1-2)}(\lambda)/(n_1-2)! & f^{(n_1-1)}(\lambda)/(n_1-1)! \\ 0 & f(\lambda) & f'(\lambda) & \cdots & f^{(n_1-3)}(\lambda)/(n_1-3)! & f^{(n_1-2)}(\lambda)/(n_1-2)! \\ 0 & 0 & f(\lambda) & \cdots & f^{(n_1-4)}(\lambda)/(n_1-4)! & f^{(n_1-3)}(\lambda)/(n_1-3)! \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{bmatrix}, \tag{10-17}$$

an $n_1 \times n_1$ matrix that has f(λ) on it main diagonal, $f'(\lambda)$ on its first superdiagonal, $f''(\lambda)/2!$ on its second superdiagonal, and so on (primes denote derivatives, and $f^{(k)}$ denotes the k^{th} derivative).

Example

Calculate f(J) for the function $f(\lambda) = e^{\lambda t}$. Direct application of (10-17) produces

$$f(J) = \exp[Jt]$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}/2! & \cdots & t^{n-2}e^{\lambda t}/(n_1-2)! & t^{n-1}e^{\lambda t}/(n_1-1)! \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & t^{n-3}e^{\lambda t}/(n_1-3)! & t^{n-2}e^{\lambda t}/(n_1-2)! \\ 0 & 0 & e^{\lambda t} & \cdots & t^{n-4}e^{\lambda t}/(n_1-4)! & t^{n-3}e^{\lambda t}/(n_1-3)! \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda t} \end{bmatrix}$$

Example

Consider the matrix $A = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$

If $f(\lambda) = e^{\lambda t}$, find f(A). Note that A contains two Jordan blocks. Hence, the previous example can be applied twice, once to each block. The result is

$$f(A) = e^{At} = \left[\begin{array}{ccc|cc} e^{\lambda_1 t} & te^{\lambda_1 t} & \tfrac{1}{2}t^2 e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ \hline 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{array}\right]$$

Example From the end of Chapter 9, consider the example

$$A = \begin{bmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = P \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_3 \end{bmatrix} P^{-1},$$

where matrix P, J_1, J_2 and J_3 are

$$P = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J_3 = [0]$$

Let's compute exp(A) by using the information given above. The answer is

$$\exp(A) = P\begin{bmatrix} \exp(J_1) & & \\ & \exp(J_2) & \\ & & \exp(J_3) \end{bmatrix} P^{-1}$$

where

$$\exp(J_1) = \begin{bmatrix} e^2 & e^2 & \tfrac{1}{2}e^2 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}, \quad \exp(J_2) = \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix}, \quad \exp(J_3) = [1].$$

Let's put it all together by using MatLab to do the "heavy lifting".

```
% First enter the 6x6 transformation matrix P into MatLab
> P= [2 1 0 0 0 0;2 -1 0 0 0 0;0 0 1 2 1 0;0 0 0 -2 -1 0;0 0 0 0 1 1;0 0 0 0 1 -1];
% Enter the 6x6 Jordan canonical form
> e2 = exp(2);
>EJ = [e2 e2 e2/2 0 0 0;0 e2 e2 0 0 0;0 0 e2 0 0 0;0 0 0 e2 e2 0;0 0 0 0 e2 0;0 0 0 0 0 1];
% Calculate exp(A) =
```

$$> P*EJ*inv(P) = \begin{bmatrix} 14.7781 & -7.3891 & 14.7781 & 14.7781 & 0 & 0 \\ 7.3891 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7.3891 & 0 & 7.3891 & 7.3891 \\ 0 & 0 & 0 & 7.3891 & -7.3891 & -7.3891 \\ 0 & 0 & 0 & 0 & 4.1945 & 3.1945 \\ 0 & 0 & 0 & 0 & 3.1945 & 4.1945 \end{bmatrix}$$

```
% Calculate exp(A) by using MatLab's built-in function expm( )
> A = [3 -1 1 1 0 0;1 1 -1 -1 0 0;0 0 2 0 1 1;0 0 0 2 -1 -1;0 0 0 0 1 1;0 0 0 0 1 1];
```

$$> expm(A) = \begin{bmatrix} 14.7781 & -7.3891 & 14.7781 & 14.7781 & 0 & 0 \\ 7.3891 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7.3891 & 0 & 7.3891 & 7.3891 \\ 0 & 0 & 0 & 7.3891 & -7.3891 & -7.3891 \\ 0 & 0 & 0 & 0 & 4.1945 & 3.1945 \\ 0 & 0 & 0 & 0 & 3.1945 & 4.1945 \end{bmatrix}$$

```
% The results using the Jordan form are the same as those obtained by using MatLab's expm
% function!
```

Example

For the A matrix considered in the last example, we can calculate sin(A). In terms of the transformation matrix P and Jordan form given in the previous example, the answer is sin(A) =

P·sin(Jordan Form of A)·P[1]. The sine of the Jordan form is

$$\sin\left(\begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix}\right) = \begin{bmatrix} \sin(2) & \cos(2) & -\tfrac{1}{2}\sin(2) & 0 & 0 & 0 \\ 0 & \sin(2) & \cos(2) & 0 & 0 & 0 \\ 0 & 0 & \sin(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(2) & \cos(2) & 0 \\ 0 & 0 & 0 & 0 & \sin(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin(0) \end{bmatrix}$$

A final numerical result can be obtained by using MatLab to do the messy work.

```
% Enter the 6x6 matrix P into MatLab
>P = [2 1 0 0 0 0;2 -1 0 0 0 0;0 0 1 2 1 0;0 0 0 -2 -1 0;0 0 0 0 1 1;0 0 0 0 1 -1];
% Enter the 6x6 matrix sin[Jordan Form] (i.e., sin of Jordan Canonical Form)into MatLab
>s = sin(2);
>c = cos(2);
>SinJ = [s c -s/2 0 0 0;0 s c 0 0 0;0 0 s 0 0 0;0 0 0 s c 0;0 0 0 0 s 0;0 0 0 0 0 0];
% Calculate sin(A)
>sinA = P*SinJ*inv(P)
```

$$\text{SinA} = \begin{bmatrix} 0.4932 & 0.4161 & -1.3254 & -1.3254 & 0 & 0 \\ -0.4161 & 1.3254 & -0.4932 & -0.4932 & 0 & 0 \\ 0 & 0 & 0.9093 & 0 & -0.4161 & -0.4161 \\ 0 & 0 & 0 & 0.9093 & 0.4161 & 0.4161 \\ 0 & 0 & 0 & 0 & 0.4546 & 0.4546 \\ 0 & 0 & 0 & 0 & 0.4546 & 0.4546 \end{bmatrix}$$

```
% To verify this result, Let's calculate sin(A) by using MatLab's built-in functions to compute SinA =
% imag(expm(i*A))
>SinA = imag(expm(i*A))
```

$$\text{SinA} = \begin{bmatrix} 0.4932 & 0.4161 & -1.3254 & -1.3254 & 0 & 0 \\ -0.4161 & 1.3254 & -0.4932 & -0.4932 & 0 & 0 \\ 0 & 0 & 0.9093 & 0 & -0.4161 & -0.4161 \\ 0 & 0 & 0 & 0.9093 & 0.4161 & 0.4161 \\ 0 & 0 & 0 & 0 & 0.4546 & 0.4546 \\ 0 & 0 & 0 & 0 & 0.4546 & 0.4546 \end{bmatrix}$$

The results are the same!

At the beginning of this chapter, we argued that f(A) can be represented by a series of matrices if $\|A\|_2$ is in the region of convergence of (10-1). Now, we get a little more sophisticated. We argue that the series for f(A) converges if the eigenvalues of A lie in the region of convergence of (10-1).

Theorem 10-3

If f(z) has a power series representation

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{10-18}$$

in an open disk $|z| < R$ containing the eigenvalues of A, then

$$f(A) = \sum_{k=0}^{\infty} c_k A^k \tag{10-19}$$

Proof: We prove this theorem for n×n matrices that are similar to a diagonal matrix (the more general case follows by adapting this proof to the Jordan form of A). Let transformation matrix P diagonalize A; that is, let D = diag(λ_1, ... , λ_n) = $P^{-1}AP$. By Theorem 10-2 it follows that

$$\begin{aligned} f(A) &= P\,\mathrm{diag}(f(\lambda_1),\ \cdots,\ f(\lambda_n))\,P^{-1} = P\,\mathrm{diag}(\sum_{k=0}^{\infty} c_k \lambda_1^k,\ \cdots,\ \sum_{k=0}^{\infty} c_k \lambda_n^k)\,P^{-1} \\ &= P\left(\sum_{k=0}^{\infty} c_k D^k\right)P^{-1} = \sum_{k=0}^{\infty} c_k (PDP^{-1})^k = \sum_{k=0}^{\infty} c_k A^k \ . \end{aligned} \tag{10-20}$$

♥

If (10-18) diverges when evaluated at the eigenvalue λ_i (as would be the case if $|\lambda_i| > R$), then series (10-19) diverges. Hence, if one (or more) eigenvalue falls outside of $|z| < R$, then (10-19) diverges.