Chapter 6

$A\vec{X} = \vec{b}$: The Minimum Norm Solution and the Least-Square-Error Problem

Like the previous chapter, this chapter deals with the linear algebraic equation problem $A\vec{X} = \vec{b}$. However, in this chapter, we impose additional structure in order to obtain specialized, but important, results. First, we consider **Problem #1**: $\vec{b}$ is in the range of A. For $A\vec{X} = \vec{b}$, either a unique solution exists *or* an infinite number of solutions exist. We want to find the solution with minimum norm. The minimum norm solution always exists, and it is unique. Problem #1 is called the *minimum norm problem*. Next, we consider **Problem #2:** $\vec{b}$ **is not** in the range of A so that $A\vec{X} = \vec{b}$ has no solution. Of all the vectors $\vec{X}$ that minimize $\|A\vec{X} - \vec{b}\|_2$, we want to find the one with minimum norm. Problem #2 is called the *minimum norm, least-square-error problem*. Its solution always exists and it is unique.

It should be obvious that Problems #1 and #2 are special cases of **a more general problem.** That is, given $A\vec{X} = \vec{b}$ (with no conditions placed on $\vec{b}$), we want to find *the* $\vec{X}$ which simultaneously minimizes both $\|A\vec{X} - \vec{b}\|_2$ and $\|\vec{X}\|_2$. Such an $\vec{X}$ always exists, it is always unique, and it is linearly related to $\vec{b}$. Symbolically, we write

$$\vec{X} = A^{+}\vec{b}, \tag{6-1}$$

where A^{+} denotes a linear operator called the *pseudo inverse* of A (yes, if A^{-1} exists, then $A^{+} = A^{-1}$ and $\vec{X} = A^{-1}\vec{b}$). For Problem #1, $\vec{X} = A^{+}\vec{b}$ will be *a* solution of $A\vec{X} = \vec{b}$, and it will be *the* "shortest length" (minimum norm) solution. For Problem #2, $\vec{X} = A^{+}\vec{b}$ simultaneously minimizes both $\|A\vec{X} - \vec{b}\|_2$ and $\|\vec{X}\|_2$ even though it does not satisfy $A\vec{X} = \vec{b}$ (which has no solution for Problem #2 where $\vec{b} \notin R(A)$).

<u>Problem #1: The Minimum Norm Problem</u>

In this section we consider the case where $\vec{b} \in R(A)$. The system

$$A\vec{X} = \vec{b} \tag{6-2}$$

has a unique solution *or* it has and infinite number of solutions as described by (5-7). *If* (6-2) has a unique solution, then this solution is the minimum norm solution, by default. *If* (6-2) has an infinite number of solutions, then we must find the solution with the smallest norm. In either case, the minimum norm solution is unique, and it is characterized as being orthogonal to K(A), as shown in what follows.

Each solution $\vec{X}$ of (6-2) can be *uniquely* decomposed into

$$\vec{X} = \vec{X}_{K^\perp} \oplus \vec{X}_K\,, \tag{6-3}$$

where

$$\begin{aligned} &\vec{X}_{K^\perp} \in K(A)^\perp = R(A^*) \\ &\vec{X}_K \;\;\in K(A) \end{aligned}. \tag{6-4}$$

However

$$A\vec{X} = A(\vec{X}_{K^\perp} \oplus \vec{X}_K) = A\vec{X}_{K^\perp} = \vec{b}\,, \tag{6-5}$$

so $\vec{X}_{K^\perp} \in K(A)^\perp$ is the only part of $\vec{X}$ that is significant in generating $\vec{b}$.

Theorem 6.1

In the decomposition (6.3), there is only one vector $\vec{X}_{K^\perp} \in K(A)^\perp$ that is common to all solutions of $A\vec{X} = \vec{b}$. Equivalently, there is a unique vector $\vec{X}_{K^\perp} \in K(A)^\perp$ for which $A\vec{X}_{K^\perp} = \vec{b}$.

Proof: To show this, assume there are two, and arrive at a contradiction. We assume the existence of $\vec{X}_{K^\perp} \in K(A)^\perp$ and $\vec{Y}_{K^\perp} \in K(A)^\perp$ such that $A\vec{X}_{K^\perp} = \vec{b}$ and $A\vec{Y}_{K^\perp} = \vec{b}$. Then, simple subtraction leads to the conclusion that $A(\vec{X}_{K^\perp} - \vec{Y}_{K^\perp}) = \vec{0}$, or the difference $\vec{X}_{K^\perp} - \vec{Y}_{K^\perp} \in K(A)$. This contradiction (as we know, $\vec{X}_{K^\perp} - \vec{Y}_{K^\perp} \in K(A)^\perp$) leads to the conclusion that there is only one $\vec{X}_{K^\perp} \in K(A)^\perp$ that is common to all solutions of $A\vec{X} = \vec{b}$ (each solution has a decomposition of the form (6.3) where a common $\vec{X}_{K^\perp}$ is used).♥

Theorem 6.2

The unique solution $\vec{X}_{K^\perp} \in K(A)^\perp$ is the *minimum norm* solution of $A\vec{X} = \vec{b}$. That is,

$$\left\|\vec{X}_{K^\perp}\right\|_2 < \left\|\vec{X}\right\|_2 , \tag{6-6}$$

where $\vec{X}$ is any other (*i.e.*, $\vec{X} \neq \vec{X}_{K^\perp}$) solution of $A\vec{X} = \vec{b}$.

Proof:

Let $\vec{X}$, $\vec{X} \neq \vec{X}_{K^\perp}$, be any solution of $A\vec{X} = \vec{b}$. As shown by (6-3), we can write the decomposition

$$\vec{X} = \vec{X}_{K^\perp} \oplus \vec{X}_K , \tag{6-7}$$

where $\vec{X}_K \in K(A)$. Clearly, we have

$$\begin{aligned}\left\|\vec{X}\right\|_2^2 &= \left\|\vec{X}_{K^\perp} \oplus \vec{X}_K\right\|_2^2 = \left\langle \vec{X}_{K^\perp} \oplus \vec{X}_K,\ \vec{X}_{K^\perp} \oplus \vec{X}_K \right\rangle \\ &= \left\langle \vec{X}_{K^\perp} ,\ \vec{X}_{K^\perp} \right\rangle + \left\langle \vec{X}_K,\ \vec{X}_K \right\rangle \quad \text{(due to orthogonality of vectors)} \\ &= \left\|\vec{X}_{K^\perp}\right\|_2^2 + \left\|\vec{X}_K\right\|_2^2 ,\end{aligned} \tag{6-8}$$

so that $\left\|\vec{X}_{K^\perp}\right\|_2 < \left\|\vec{X}\right\|_2$ as claimed.♥

When viewed geometrically, Theorem 6.2 is obvious. As shown by Figure 6.1, the solution set for $A\vec{X} = \vec{b}$ is a linear variety, a simple translation of K(A). Figure 6.1 shows an

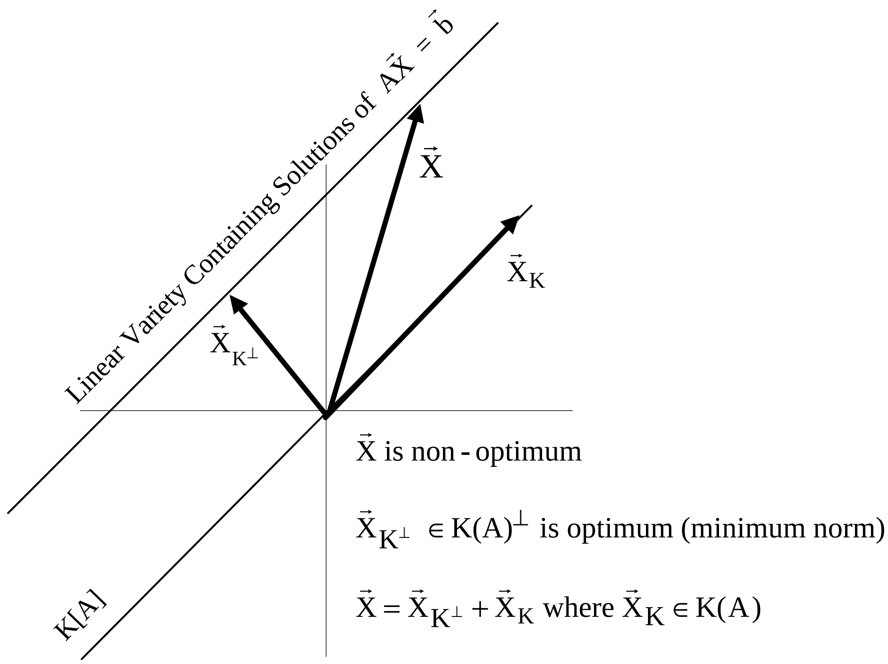


Figure 6-1: The solution set is a linear variety. The minimum norm solution is orthogonal to K(A).

arbitrary solution $\vec{X}$, and it shows the "optimum", minimum norm, solution $\vec{X}_{K^\perp}$. Clearly, the solution becomes "optimum" when it is exactly orthogonal to K(A) (*i.e.*, it is in $K(A)^\perp$). We have solved Problem #1.

<u>The Pseudo Inverse A^+</u>

As shown by Theorems 6.1 and 6.2, when $\vec{b} \in R(A)$, a unique minimum norm solution to $A\vec{X} = \vec{b}$ always exists. And, this solution is in $R(A^*) = K(A)^\perp$. Hence we have a mapping from $\vec{b} \in R(A)$ *back to* $\vec{X} \in R(A^*)$ (at this point, it might be a good idea to study Fig. 5.1 once again). It is not difficult to show that this mapping is linear, one-to-one and onto. We denote this mapping by

$$A^+ : R(A) \rightarrow R(A^*), \tag{6-9}$$

and we write $\vec{X} = A^+\vec{b}$, for $\vec{b} \in R(A)$ and $\vec{X} \in R(A^*)$. Finally, our unique minimum norm solution to the $A\vec{X} = \vec{b}$, $\vec{b} \in R(A)$, problem (*i.e.*, "Problem #1") is denoted symbolically by $\vec{X}_{K^\perp} = A^+\vec{b}$.

As a subspace of **U**, $R(A^*)$ is a vector space in its own right. Every $\vec{X}$ in vector space $R(A^*) \subset$ **U** gets mapped by matrix A to something in vector space $R(A) \subset$ **V**. By restricting the domain of A to be just $R(A^*)$ (instead of the whole **U**), we have a mapping with domain $R(A^*)$ and co-domain R(A). Symbolically, we denote this mapping by

$$A\big|_{R[A^*]} : R(A^*) \rightarrow R(A), \tag{6-10}$$

and we call it the *restriction* of A to $R(A^*)$. The mapping (6-10) is linear, one-to-one, and onto (even though A : **U** → **V** may not be one-to-one *or* onto). More importantly, the inverse of mapping (6-10) is the mapping (6-9). As is characteristic of an operator and its inverse, we have

$$\begin{aligned} &A\big|_{R[A^*]} A^+ \vec{b} = \vec{b} \text{ for all } \vec{b} \in R(A) \\ &A^+ A\big|_{R[A^*]} \vec{X} = \vec{X} \text{ for all } \vec{X} \in R(A^*) \end{aligned} , \tag{6-11}$$

as expected. Finally, the relationship between (6-9) and (6-10) is illustrated by Fig. 6-2.

As defined so far, the domain of A^+ is *only* $R(A) \subset$ **V**. This is adequate for our discussion of "Problem #1"-type problems where $\vec{b} \in R(A)$. However, for our discussion of "Problem #2" (and the more general optimization problem, discussed in the paragraph containing (6-1), where $\vec{b}$ $\in$ **V**), we must extend the domain of A^+ to be all of **V** $= R(A) \oplus K(A^*)$. A^+ is already defined on

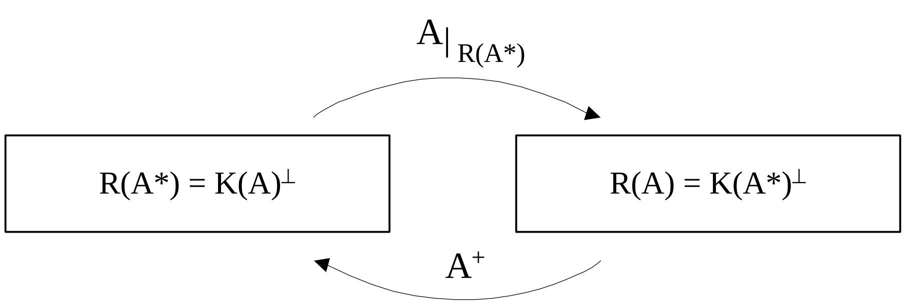


Figure 6-2: Restricted to $R(A^*) = K(A)^{\perp}$, A is one-to-one and onto, and it has the inverse A^+.

R(A); we need to define the operator on K(A^*). We do this in a very simple manner. We *define*

$$A^+\vec{b} = \vec{0}, \;\; \vec{b} \in K(A^*). \tag{6-12}$$

With the extension offered by (6-12), we have the entire vector space **V** as the domain of A^+. The operations performed by A and A^+ are illustrated by Fig. 6.3. When defined in this manner, A^+ is called the *Moore-Penrose pseudo inverse of A* (or simply the *pseudo inverse*).

A number of important observations can be made from inspection of Fig. 6-3. Note that

(1) $AA^+\vec{X} = \vec{0}$ for $\vec{X} \in K(A^*) = R(A)^\perp$,

(2) $AA^+\vec{X} = \vec{X}$ for $\vec{X} \in K(A^*)^\perp = R(A)$,

(3) $A^+A\vec{X} = \vec{0}$ for $\vec{X} \in K(A) = R(A^*)^\perp$, (6-13)

(4) $A^+A\vec{X} = \vec{X}$ for $\vec{X} \in R(A^*) = K(A)^\perp$, and

(5) $R(A^*) = R(A^+)$ and $K(A^*) = K(A^+)$ (however, $A^* \neq A^+$).

Note that (1) and (2) of (6-13) imply that AA^+ is the orthogonal projection of **V** onto R(A). Also,

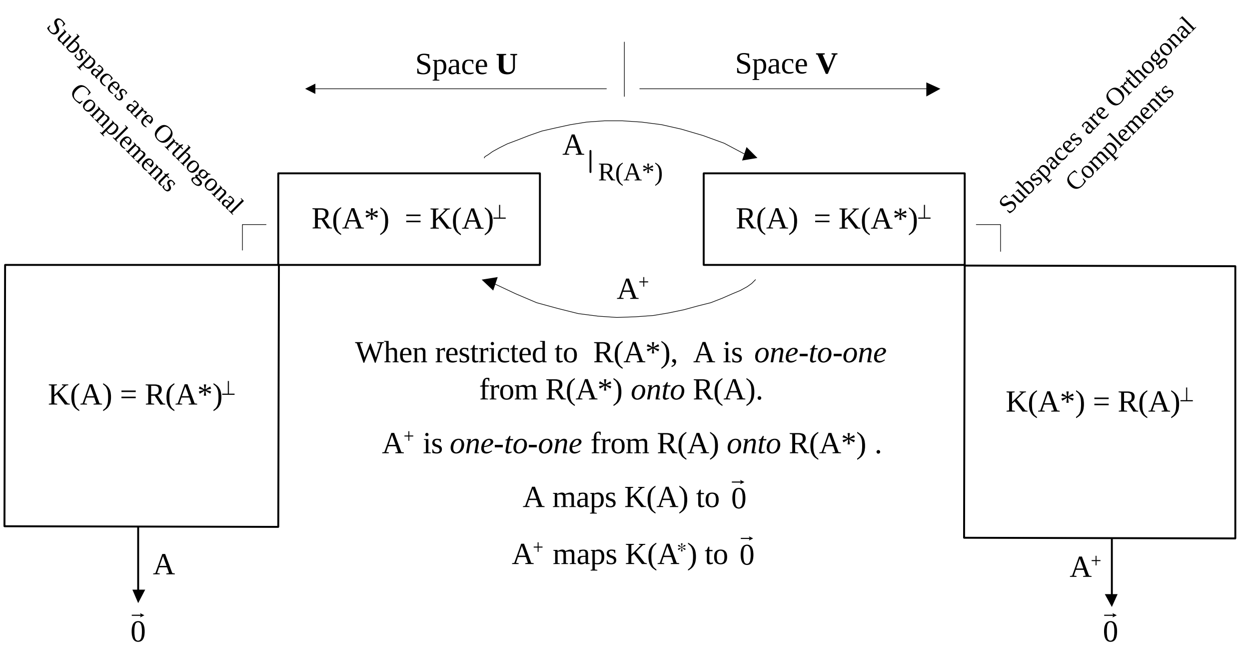
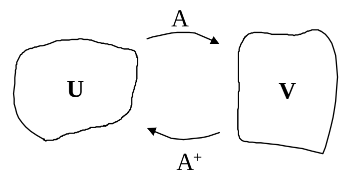


Figure 6-3: How A and A^+ map the various parts of **U** and **V.**

(3) and (4) of (6-13) imply that A^+A is the orthogonal projection of **U** onto $R(A^*) = R(A^+)$. Finally, when A^{-1} exists we have $A^+ = A^{-1}$.

Original Moore and Penrose Definitions of the Pseudo Inverse



In 1935, Moore defined A^+ as the unique n×m matrix for which

1) AA^+ is the orthogonal projection of **V** onto $R(A) = K(A^*)^\perp$, and (6-14)

2) A^+A is the orthogonal projection of **U** onto $R(A^*) = K(A)^\perp$.

In 1955, Penrose gave a purely algebraic definition of the pseudo inverse. He said that A^+ is the unique n×m matrix satisfying

3) $AA^+A = A$,

4) $A^+AA^+ = A^+$, (6-15)

5) AA^+ and A^+A are Hermitian.

Of course Moore's and Penrose's definitions are equivalent, and they agree with the geometric description we gave in the previous subsection.

While not very inspiring, Penrose's algebraic definition makes it simple to check if a given matrix is A^+. Once we compute a candidate for A^+, we can verify our results by checking if our candidate satisfies Penrose's conditions. In addition, Penrose's definition leads to some simple and useful results as shown by the next few theorems.

Theorem 6.3

$$(A^+)^+ = A^{++} = A \tag{6-16}$$

Proof: The result follows by direct substitution into (6-15).

Theorem 6.4

$$(A^*)^+ = (A^+)^* \text{ or } A^{*+} = A^{+*} . \tag{6-17}$$

That is, the order of the + and $*$ is not important!

Proof: We claim that the pseudo inverse of A^* is A^{+*}. Let's use (6-15) to check this out!

3′) $A^*A^{+*}A^* = A^*(AA^+)^* = ((AA^+)A)^* = (AA^+A)^* = A^*$ so that 3) holds!

4′) $A^{+*}A^*A^{+*} = A^{+*}(A^+A)^* = ((A^+A)A^+)^* = (A^+AA^+)^* = A^{+*}$ so that 4) holds!

5′) $A^{+*}A^* = (AA^+)^* = ((AA^+)^*)^* = (A^{+*}A^*)^*$ so $A^{+*}A^*$ is Hermitian! So is A^*A^{+*} !!

Theorem 6.5

Suppose A is Hermitian. Then A^+ is Hermitian.

Proof: $(A^+)^* = (A^*)^+ = A^+$.

Given matrices A and B, one might ask if $(AB)^+ \stackrel{?}{=} B^+A^+$ in general. The answer is **NO!** It is not difficult to find some matrices that illustrate this. Hence, a well known and venerable property of inverses (*i.e.*, that $(AB)^{-1} = B^{-1}A^{-1}$) **does not** carry over to psuedo inverses.

Computing the Pseudo Inverse A^+

There are three cases that need to be discussed when one tries to find A^+. The first case is when rank(A) < min(m,n) so that A does not have full rank. The second case is when m×n matrix A has full row rank (rank(A) = m). The third case is when A has full column rank (rank(A) = n).

Case #1: r = Rank(A) < Min(m,n)

In this case, K(A) contains non-zero vectors, and the system $A\vec{X} = \vec{b}$ has an infinite number of solutions. For this case, a *simple* "plug-in-the-variables" formula for A^+ does not exist. However, the pseudo inverse can be calculated by using the basic ideas illustrated by Fig. 6.3.

Suppose m×n matrix A has rank r. Let $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_r$ be a basis of $R(A^*)$ and $\vec{Y}_1, \vec{Y}_2, \ldots,$ $\vec{Y}_{m-r}$ be a basis of $K(A^*)$. From (6-11) and Fig. 6.3, we see

$$\begin{aligned} A^+\left[A\vec{X}_1 \,\vdots\, \cdots \,\vdots\, A\vec{X}_r \,\vdots\, \vec{Y}_1 \,\vdots\, \cdots \,\vdots\, \vec{Y}_{m-r}\right] &= \left[A^+A\vec{X}_1 \,\vdots\, \cdots \,\vdots\, A^+A\vec{X}_r \,\vdots\, A^+\vec{Y}_1 \,\vdots\, \cdots \,\vdots\, A^+\vec{Y}_{m-r}\right] \\ &= \left[\vec{X}_1 \,\vdots\, \cdots \,\vdots\, \vec{X}_r \,\vdots\, \vec{0} \,\vdots\, \cdots \,\vdots\, \vec{0}\right]. \end{aligned} \tag{6-18}$$

Now, note that $A\vec{X}_1, \ldots, A\vec{X}_r$ is a basis for R(A). Since $\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_{m-r}$ is a basis of $K(A^*)$ =

$R(A)^{\perp}$, we see that $m \times m$ matrix $\left[A\vec{X}_1 \,\vdots\, \cdots \,\vdots\, A\vec{X}_r \,\vdots\, \vec{Y}_1 \,\vdots\, \cdots \,\vdots\, \vec{Y}_{m-r}\right]$ is nonsingular. From (6-18) we have

$$A^{+} = [\vec{X}_1 \,\vdots\, \cdots \,\vdots\, \vec{X}_r \,\vdots\, \underbrace{\vec{0} \,\vdots\, \cdots \,\vdots\, \vec{0}}_{\substack{m-r \text{ zero}\\ \text{columns}}}]\left[A\vec{X}_1 \,\vdots\, \cdots \,\vdots\, A\vec{X}_r \,\vdots\, \vec{Y}_1 \,\vdots\, \cdots \,\vdots\, \vec{Y}_{m-r}\right]^{-1} \tag{6-19}$$

While not simple, Equation (6-19) is "useable" in many cases, especially when m and n are small integers.

Example

```
% Pseudo Inverse Example
% Enter 4x3 matrix A.
A = [1 1 2; 0 2 2; 1 0 1; 1 0 1]

A = 1     1     2
    0     2     2
    1     0     1
    1     0     1
% Have MatLab Calculate an Orthonormal Basis of R(A*)
X = orth(A')
X = 0.3052   -0.7573
    0.5033    0.6430
    0.8084   -0.1144
% Have MatLab Calculate an Orthonormal Basis of K(A*)
Y = null(A')

Y = 0.7559         0
   -0.3780    0.0000
   -0.3780   -0.7071
   -0.3780    0.7071

% Use (6-19) of class notes to calculate the pseudo inverse
pseudo = [  X   [0 0 0]'   [0 0 0]'  ]*inv([A*X Y])
pseudo = 0.1429   -0.2381    0.2619    0.2619
         0.0000    0.3333   -0.1667   -0.1667
         0.1429    0.0952    0.0952    0.0952

% Same as that produced by MatLab's pinv function?
pinv(A)
ans = 0.1429   -0.2381    0.2619    0.2619
      0.0000    0.3333   -0.1667   -0.1667
      0.1429    0.0952    0.0952    0.0952

% YES! YES!
```

Case #2: r = Rank(A) = m (A has full row rank)

For this case, we have m ≤ n, the columns of A *may or may not* be dependent, and $A\vec{X} = \vec{b}$ *may or may not* have an infinite number of solutions. For this case, n×m matrix A* has full column rank and (6-19) becomes

$$A^{+} = A^{*}\left(AA^{*}\right)^{-1}, \tag{6-20}$$

a simple formula.

Case #3: Rank(A) = n (A has full column rank)

For this case, we have n ≤ m, the columns of A *are* independent. For this case, the n×n matrix A^*A is nonsingular (why?). On the left, we multiply the equation $A\vec{X} = \vec{b}$ by A^* to obtain $A^*A\vec{X} = A^*\vec{b}$, and this leads to

$$\vec{X}_{K^{\perp}} = (A^{*}A)^{-1}A^{*}\vec{b} \tag{6-21}$$

When A has full column rank, the pseudo inverse is

$$A^{+} = (A^{*}A)^{-1}A^{*}, \tag{6-22}$$

a simple formula.

Example 6-1 $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Note that rank(A) = 2 and nullity(A) = 1. By inspection, the general solution is

$$\vec{X} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now, find the minimum norm solution. We do this two ways. Since this problem is so simple, lets compute

$$\|\vec{X}\|_2^2 = (2+\alpha)^2 + 1^2 + \alpha^2 .$$

The minimum value of this is found by computing

$$\frac{d}{d\alpha}\|\vec{X}\|_2^2 = 2(2+\alpha) + 2\alpha = 0$$

which yields α = -1. Hence, the minimum norm solution is

$$\vec{X}_{K^\perp} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{6-23}$$

a vector that is orthogonal to K(A) as required (please check orthogonality). This same result can be computed by using (6-20) since, as outlined above, special case #2 applies here. First, we compute

$$AA^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ (AA^*)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, we use these results with (6-20) to obtain

$$\vec{X}_{K^\perp} = A^+\vec{b} = A^*(AA^*)^{-1}\vec{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

the same results as (6-23)!

MatLab's Backslash (\ ) and Pinv Functions

A unique solution of $A\vec{X} = \vec{b}$ exists if $\vec{b} \in R(A)$ and nullity(A) = 0. It can be found by using MatLab's backslash (\ ) function. The syntax is `X = A\b`; MatLab calls their backslash *matrix left division*.

Let r = rank(A) and nullity(A) = n - r ≥ 1. For if $\vec{b} \in R(A)$, the complete solution can be described in terms of n - r independent parameters, as shown by (5-7). These independent parameters can be chosen to yield a solution $\vec{X}$ that contains *at least* n - r zero components. Such a solution is generated by the MatLab syntax `X = A\b`. So, if a unique solution exists, MatLab's backslash operator finds it; otherwise, the backslash operator finds the solution that contains the maximum number of zeros (in general, this maximum-number-of-zeros solution will **not** be $\vec{X}_{K^\perp}$, the minimum norm solution).

MatLab can also find the optimum, minimum norm, solution $\vec{X}_{K^\perp}$. The MatLab syntax for this is `X = pinv(A)∗b`. If a unique solution to $A\vec{X} = \vec{b}$ exists, then you can find it using this syntax (but, for this case, the backslash operator is a better method). If $A\vec{X} = \vec{b}$ has an infinite number of solutions, then `X = pinv(A)∗b` finds the optimum, minimum norm solution. MatLab calculates the pseudo inverse by using the singular value decomposition of A, a method that we will discuss in Chapter 8.

Example 6-2 (Continuation of Example 6-1)

We apply MatLab to the problem given by Example 6-1. The MatLab diary file follows.

```
% use the same A and b as was used in Example 6-1
A = [1 0 1;0 1 0];
b = [2;1];

% find the solution with the maximum number of zero components
X = A\b
X =
     2
     1
     0
norm(X)
```

```
ans = 2.2361
%
% find the optimum, minimum norm, solution
Y = pinv(A)*b
Y =
    1.0000
    1.0000
    1.0000
% solution is the same as that found in example 6-1!
norm(Y)
ans = 1.7321
% the "minimum norm" solution has a smaller norm
% then the "maximum-number-of-zeros" solution.
```

Problem #2: The Minimum Norm, Least-Square-Error Problem (Or: What Can We Do With Inconsistent Linear Equations?)

There are many practical problems that result, in one way or another, in an inconsistent system $A\vec{X} = \vec{b}$ of equations (*i.e.*, $\vec{b} \notin R(A)$). A solution does not exist in this case. However, we can "least-squares fit" a vector $\vec{X}$ to an inconsistent system. That is, we can consider the *least-squares error* problem of calculating an $\vec{X}$ that minimizes the error $\left\| A\vec{X} - \vec{b} \right\|_2$ (equivalent to minimizing the *square error* $\left\| A\vec{X} - \vec{b} \right\|_2^2$). As should be obvious (or, with just a little thought!), such an $\vec{X}$ is not always unique. Hence, we find the *minimum norm* vector that minimizes the error $\left\| A\vec{X} - \vec{b} \right\|_2$ (often, the error vector $A\vec{X}$ - $\vec{b}$ is called the *residual*). As it turns out, this minimum norm solution to the least-squares error problem is unique, and it is denoted as $\vec{X} = A^+\vec{b}$, where A^+ is the pseudo inverse of A. Furthermore, with MatLab's pinv function, we can solve this *minimum-norm, least-square-error* problem (as it is called in the literature). However, before we jump too far into this problem, we must discuss *orthogonal projections* and *projection operators*.

The Orthogonal Projection of One Vector on Another

Let's start out with a simple, one dimensional, projection problem. Suppose we have two vectors, $\vec{X}_1$ and $\vec{X}_2$, and we want to find the "orthogonal projection", denoted here as $\vec{X}_p$, of vector $\vec{X}_2$ onto vector $\vec{X}_1$. As depicted by Figure 6-4, the "orthogonal projection" is co-linear with $\vec{X}_1$. And, the error vector $\vec{X}_2$ - $\vec{X}_p$ is orthogonal to $\vec{X}_1$.

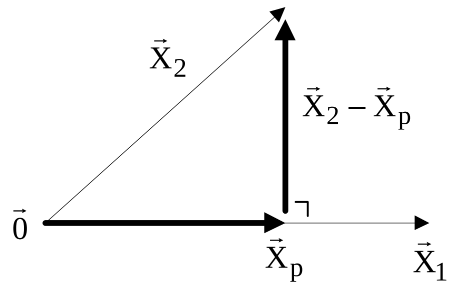


Figure 6-4: Orthogonal projection of $\vec{X}_2$ onto $\vec{X}_1$

We can find a formula for $\vec{X}_p$. Since the error must be orthogonal to $\vec{X}_1$, we can write

$$0 = \langle \vec{X}_2 - \vec{X}_p, \vec{X}_1 \rangle = \langle \vec{X}_2 - \alpha\vec{X}_1, \vec{X}_1 \rangle = \langle \vec{X}_2, \vec{X}_1 \rangle - \alpha \langle \vec{X}_1, \vec{X}_1 \rangle \tag{6-24}$$

which leads to

$$\alpha = \frac{\langle \vec{X}_2, \vec{X}_1 \rangle}{\|\vec{X}_1\|_2^2} \tag{6-25}$$

and

$$\vec{X}_p = \frac{\langle \vec{X}_2, \vec{X}_1 \rangle}{\|\vec{X}_1\|_2^2} \vec{X}_1 = \left\langle \vec{X}_2, \frac{\vec{X}_1}{\|\vec{X}_1\|_2} \right\rangle \frac{\vec{X}_1}{\|\vec{X}_1\|_2}. \tag{6-26}$$

Hence, $\vec{X}_p$ is the component of $\vec{X}_2$ in the $\vec{X}_1$ direction; $\vec{X}_2$ - $\vec{X}_p$ is the component of $\vec{X}_2$ in a direction that is perpendicular to $\vec{X}_1$. In the next section, these ideas are generalized to the projection of a vector on an arbitrary subspace.

Orthogonal Projections

Let W be a subspace of n-dimensional vector space **U** (the dimensionality of W is not important in this discussion). An n×n matrix P is said to represent an *orthogonal projection operator* on W (more simply, P is said to be an *orthogonal projection* on W) if

a) R(P) = W

b) $P^2 = P$ (*i.e.*, P is *idempotent*) (6-27)

c) P* = P (*i.e.*, P is *Hermitian*)

We can obtain some simple results from consideration of definition (6-27). **First**, from a) we have $P\vec{X} \in W$ for all $\vec{X} \in \mathbf{U}$. **Secondly**, from a) and b) we have $P\vec{X} = \vec{X}$ for all $\vec{X} \in W$. To see this, note that for $\vec{X} \in W = R(P)$ we have $P(P\vec{X} - \vec{X}) = P^2\vec{X} - P\vec{X} = P\vec{X} - P\vec{X} = \vec{0}$ so that $P\vec{X} - \vec{X} \in K(P)$. However, $P\vec{X} - \vec{X} \in R(P) = W$. The only vector that is in K(P) and R(P) simultaneously is $\vec{0}$. Hence, $P\vec{X} = \vec{X}$ for each and every $\vec{X} \in W$. The **third** observation from (6-27) is that a), b) and c) tell us that $(I - P)\vec{X} \in W^{\perp}$ for each $\vec{X} \in \mathbf{U}$. To see this, consider any $\vec{Y} \in W = R(P)$. Then, $\vec{Y} = P\vec{X}_2$ for some $\vec{X}_2 \in \mathbf{U}$. Now, take any $\vec{X}_1 \in \mathbf{U}$ and write

$$\begin{aligned}\left\langle (I-P)\vec{X}_1, \vec{Y}\right\rangle &= \left\langle (I-P)\vec{X}_1, P\vec{X}_2\right\rangle = \left\langle P^*(I-P)\vec{X}_1, \vec{X}_2\right\rangle = \left\langle P(I-P)\vec{X}_1, \vec{X}_2\right\rangle \\ &= \left\langle P\vec{X}_1 - P^2\vec{X}_1, \vec{X}_2\right\rangle = \left\langle P\vec{X}_1 - P\vec{X}_1, \vec{X}_2\right\rangle \\ &= 0\end{aligned} \qquad (6\text{-}28)$$

Hence, for all $\vec{X}_1 \in \mathbf{U}$, we have $(I - P)\vec{X}_1$ orthogonal to W. Hence, $\vec{X}$ is decomposed into a part $P\vec{X} \in W$ and a part $(I - P)\vec{X} \in W^{\perp}$. Figure 6.5 depicts this important decomposition. The **fourth** observation from (6-27) is

$$\vec{X} \in W^{\perp} \;\Rightarrow\; P\vec{X} = \vec{0}. \qquad (6\text{-}29)$$

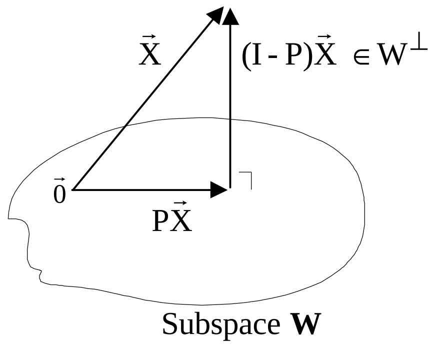


Figure 6-5: Orthogonal Projection of $\vec{\mathbf{X}}$ onto a subspace W. $\vec{\mathbf{X}}$ is decomposed into $P\vec{\mathbf{X}} \in$ W and (I - P)$\vec{\mathbf{X}} \in W^{\perp}$.

This is obvious; it follows from the facts that

$$\begin{aligned} &P\vec{X} \in W \\ &(I - P)\vec{X} = \vec{X} - P\vec{X} \in W^{\perp} \end{aligned}, \tag{6-30}$$

and if $\vec{X} \in W^{\perp}$, the only way (6-30) can be true is to have $P\vec{X} = \vec{0}$ for all $\vec{X} \in W^{\perp}$. The **fifth** observation is that P plays a role in the unique direct sum decomposition of any $\vec{X} \in$ U. Recall that $U = W \oplus W^{\perp}$, and any $\vec{X} \in$ U has the *unique* decomposition $\vec{X} = \vec{X}_W \oplus \vec{X}_{W^{\perp}}$, where $\vec{X}_W \in$ W and $\vec{X}_{W^{\perp}} \in W^{\perp}$. Note that $P\vec{X} = \vec{X}_W$ and $(I - P)\vec{X} = \vec{X}_{W^{\perp}}$. Hence, if P is the orthogonal projection operator on W, then (I - P) is the orthogonal projection operator on $W^{\perp}$.

<u>PX is the Optimal Approximation of X</u>

Of all vectors in W, $P\vec{X}$ is the "optimal" approximation of $\vec{X} \in$ **U**. Consider any $\vec{X}_\delta \in$ W and any $\vec{X} \in$ **U**, and note that $\vec{Z} \equiv P\vec{X} + \vec{X}_\delta \in$ W (think of $\vec{X}_\delta \in$ W as being a perturbation of $P\vec{X} \in$ W). We show that $\|\vec{X} - P\vec{X}\|_2 \le \|\vec{X} - \{P\vec{X} + \vec{X}_\delta\}\|_2$ for all $\vec{X} \in$ **U** and $\vec{X}_\delta \in$ W. Consider

$$\begin{aligned} \|\vec{X} - \{P\vec{X} + \vec{X}_\delta\}\|_2^2 &= \langle (\vec{X} - P\vec{X}) - \vec{X}_\delta,\ (\vec{X} - P\vec{X}) - \vec{X}_\delta \rangle \\ &= \langle \vec{X} - P\vec{X},\ \vec{X} - P\vec{X} \rangle - \langle \vec{X}_\delta,\ \vec{X} - P\vec{X} \rangle - \langle \vec{X} - P\vec{X}, \vec{X}_\delta \rangle + \langle \vec{X}_\delta, \vec{X}_\delta \rangle . \end{aligned} \tag{6-31}$$

However, the cross terms $\langle \vec{X}_\delta,\ \vec{X} - P\vec{X} \rangle$ and $\langle \vec{X} - P\vec{X}, \vec{X}_\delta \rangle$ are zero so that

$$\left\|\vec{X} - \{P\vec{X} + \vec{X}_\delta\}\right\|_2^2 = \left\|\vec{X} - P\vec{X}\right\|_2^2 + \left\|\vec{X}_\delta\right\|_2^2 . \tag{6-32}$$

From this, we conclude that

$$\left\|\vec{X} - P\vec{X}\right\|_2^2 \le \left\|\vec{X} - \{P\vec{X} + \vec{X}_\delta\}\right\|_2^2 = \left\|\vec{X} - P\vec{X}\right\|_2^2 + \left\|\vec{X}_\delta\right\|_2^2 \tag{6-33}$$

for all $\vec{X}_\delta \in$ W and $\vec{X} \in$ **U**. Of all the vectors in subspace W, $P\vec{X} \in$ W comes "closest" (in the 2-norm sense) to $\vec{X} \in$ **U.**

<u>Projection Operator on W is Unique</u>

The orthogonal projection on a subspace is *unique*. Assume, for the moment, that there are n×n matrices P_1 and P_2 with the properties given by (6-27). Use these to write

$$\begin{aligned}\left\|(P_1 - P_2)\vec{X}\right\|_2^2 &= \vec{X}^*(P_1 - P_2)^*(P_1 - P_2)\vec{X} = \left(P_1\vec{X}\right)^*(P_1 - P_2)\vec{X} - \left(P_2\vec{X}\right)^*(P_1 - P_2)\vec{X} \\ &= \left(P_1\vec{X}\right)^*\left(\{P_1 - I\} - \{P_2 - I\}\right)\vec{X} \;-\; \left(P_2\vec{X}\right)^*\left(\{P_1 - I\} - \{P_2 - I\}\right)\vec{X}\end{aligned} \tag{6-34}$$

However both $P_1\vec{X}$ and $P_2\vec{X}$ are orthogonal to both $(P_1 - I)\vec{X}$ and $(P_2 - I)\vec{X}$. This leads to the conclusion that

$$\left\|(P_1 - P_2)\vec{X}\right\|_2 = 0 \tag{6-35}$$

for all $\vec{X} \in$ **U**, a result that requires $P_1 = P_2$. Hence, given subspace W, the orthogonal projection operator on W is unique, as originally claimed.

How to "Make" an Orthogonal Projection Operator

Let $\vec{v}_1$, $\vec{v}_2$, ... , $\vec{v}_k$ be n×1 orthonormal vectors that span k-dimensional subspace W of n-dimensional vector space **U**. Use these vectors as columns in the n×k matrix

$$Q \equiv \left[\vec{v}_1 \;\vdots\; \vec{v}_2 \;\vdots\; \cdots \;\vdots\; \vec{v}_k\right]. \tag{6-36}$$

We claim that n×n matrix

$$P = Q\,Q^* \tag{6-37}$$

is the unique orthogonal projection onto subspace W. To show this, note that for any vector $\vec{X} \in$ **U** we have

$$P\vec{X} = Q\,Q^*\,\vec{X} = \left[\vec{v}_1 \;\vdots\; \vec{v}_2 \;\vdots\; \cdots \;\vdots\; \vec{v}_k\right]\begin{bmatrix} \vec{v}_1^* \\ \hdashline \vec{v}_2^* \\ \hdashline \vdots \\ \hdashline \vec{v}_k^* \end{bmatrix}\vec{X} = \left[\vec{v}_1 \;\vdots\; \vec{v}_2 \;\vdots\; \cdots \;\vdots\; \vec{v}_k\right]\begin{bmatrix} \vec{v}_1^*\,\vec{X} \\ \hdashline \vec{v}_2^*\,\vec{X} \\ \hdashline \vdots \\ \hdashline \vec{v}_k^*\,\vec{X} \end{bmatrix} \tag{6-38}$$

Now, $P\vec{X}$ is a linear combination of the orthonormal basis $\vec{v}_1$, $\vec{v}_2$, ... , $\vec{v}_k$. As $\vec{X}$ ranges over all of **U**, the vector $P\vec{X}$ ranges over all of W (since $P\vec{v}_j = \vec{v}_j$, $1 \le j \le k$, and $\vec{v}_1$, ... , $\vec{v}_k$ span W). As a result, R(P) = W; this is the first requirement of (6-27). The second requirement of (6-27) is met by realizing that

$$P^2 = (QQ^*)(QQ^*) = Q\ I\ Q^* = P, \tag{6-39}$$

since Q*Q is an k×k identity matrix I. Finally, the third requirement of (6-27) is obvious, and P =

QQ* is the unique orthogonal projection on the subspace W.

Special Case: One Dimensional Subspace W

Consider the one dimensional case W = span($\vec{X}_1$), where $\vec{X}_1$ has unit length. Then (6-37) gives us n×n matrix $P = \vec{X}_1\,\vec{X}_1^*$. Let $\vec{X}_2 \notin$ W. Then the orthogonal projection of $\vec{X}_2$ onto W is

$$\vec{X}_p = P\vec{X}_2 = (\vec{X}_1\,\vec{X}_1^*)\vec{X}_2 = \vec{X}_1\,(\vec{X}_1^*\vec{X}_2) = (\vec{X}_1^*\vec{X}_2)\vec{X}_1 = \left\langle \vec{X}_2, \vec{X}_1 \right\rangle \vec{X}_1\,, \tag{6-40}$$

where we have used the fact that $\vec{X}_1^*\vec{X}_2$ is a scalar. In light of the fact that $\vec{X}_1$ has unit length, (6-40) is the same as (6-26). Now that we are knowledgeable about projection operators, we can handle the important $A\vec{X} = \vec{b}$, $\vec{b} \notin$ R(A), problem.

The Minimum-Norm, Least-Squares-Error Problem

At last, we are in a position to solve yet another ***colossal*** problem in linear algebraic equation theory. Namely, what to do with the $A\vec{X} = \vec{b}$ problem when $\vec{b} \notin$ R(A) so that no *exact* solution(s) exist.

This type of problem occurs in applications where there are more equations (constraints) then there are unknowns to be solved for. In this case, the problem is said to be *overdetermined*. Such a problem is characterized by a matrix equation $A\vec{X} = \vec{b}$, where A is m×n with m > n, and $\vec{b}$ is m×1. Clearly, the rows of A are dependent. And, $\vec{b}$ may have been obtained by making imprecise measurements, so that rank(A) ≠ rank(A ¦ $\vec{b}$), and no solution exists. Thus, not knowing what is important in a model (*i.e.*, which constraints are the most important and which can be thrown out), and an inability to precisely measure input $\vec{b}$, can lead to an inconsistent set of equations.

A "fix" for this problem is to throw out constraints (individual equations) until the modified system has a solution. However, as discussed above, it is not always clear how to accomplish this when all of the m constraints *appear* to be equally valid (ignorance keeps us from obtain a "better" model).

Instead of throwing out constraints, it is sometimes better to *do the best with what you* *have*. Often, the best approach is to find an $\vec{X}$ that satisfies

$$\left\|A\vec{X}-\vec{b}\right\|_2 = \min_{\vec{Z}\in\mathbf{U}} \left\|A\vec{Z}-\vec{b}\right\|_2 . \tag{6-41}$$

That is, we try to get $A\vec{X}$ as close as possible to $\vec{b}$. The problem of finding $\vec{X}$ that minimizes $\left\|A\vec{X}-\vec{b}\right\|_2$ is known as the *least squares problem* (since the 2-norm is used).

A solution to the least squares problem always exists. However, it is not unique, in general. To see this, let $\vec{X}$ minimize the norm of the residual; that is, let $\vec{X}$ satisfy (6-41) so that it is a solution to the least squares problem. Then, add to $\vec{X}$ any vector in K(A) to get yet another vector that minimizes the norm of the residual.

It is common to add structure to the least squares problem in order to force a unique solution. We choose the smallest norm $\vec{X}$ that minimizes $\left\|A\vec{X}-\vec{b}\right\|_2$. That is, we want the $\vec{X}$ that 1) minimizes $\left\|A\vec{X}-\vec{b}\right\|_2$, *and* 2) minimizes $\left\|\vec{X}\right\|_2$. This problem is called the *minimum-norm, least-squares problem*. For all coordinate vectors $\vec{b}$ in **V,** it is guaranteed to have a unique solution (when $\vec{b} \in$ R(A), we have a "Problem #1"-type problem - see the discussion on page 6-1). And, as shown below, from $\vec{b}$ in **V** back to the coordinate vector $\vec{X}$ in **U,** the mapping is the pseudo inverse of A. As before, we write $\vec{X} = A^+\vec{b}$ for each $\vec{b}$ in **V**. The pseudo inverse solves the general optimization problem that is discussed in the paragraph containing Equation (6-1).

To verify that $\vec{X} = A^+\vec{b}$ is the smallest norm $\vec{X}$ that minimizes $\left\|A\vec{X}-\vec{b}\right\|_2$, let's go about the process of minimizing the norm of the residual $A\vec{X}$ - $\vec{b}$. First, recall from Fig. 5-1 that U = R(A*)⊕K(A) and **V** = R(A)⊕K(A*). Hence, $\vec{X} \in$ **U** and $\vec{b} \in$ **V** have the *unique* decompositions

$$\begin{aligned}
&\vec{X} = \vec{X}_{R[A^*]} \oplus \vec{X}_{K[A]}\,, \quad \begin{Bmatrix} \vec{X}_{R[A^*]} \in R(A^*) \\ \vec{X}_{K[A]} \in K[A] \end{Bmatrix} \\
&\vec{b} = \vec{b}_{R[A]} \oplus \vec{b}_{K[A^*]}\,, \quad \begin{Bmatrix} \vec{b}_{R[A]} \in R(A) \\ \vec{b}_{K[A^*]} \in K(A^*) \end{Bmatrix} .
\end{aligned} \tag{6-42}$$

Now, we compute

$$\begin{aligned}\left\|A\vec{X}-\vec{b}\right\|_2^2 &= \left\|A\{\vec{X}_{R[A^*]}\oplus\vec{X}_{K[A]}\}-\{\vec{b}_{R[A]}\oplus\vec{b}_{K[A^*]}\}\right\|_2^2 \\ &= \left\|\{A\vec{X}_{R[A^*]}-\vec{b}_{R[A]}\}-\vec{b}_{K[A^*]}\right\|_2^2 \\ &= \left\|A\vec{X}_{R[A^*]}-\vec{b}_{R[A]}\right\|_2^2+\left\|\vec{b}_{K[A^*]}\right\|_2^2 .\end{aligned} \tag{6-43}$$

The second line of this result follows from the fact $A\vec{X}_{K[A]}=\vec{0}$. The third line follows from the orthogonality of $\{A\vec{X}_{R[A^*]}-\vec{b}_{R[A]}\}$ and $\vec{b}_{ker[A^*]}$. In minimizing (6-43), we have no control over $\vec{b}_{K[A^*]}$. However, how to choose $\vec{X}$ should be obvious. Select $\vec{X}$ = $\vec{X}_{R[A^*]}$ $\in$ R(A^*) = K(A)$^\perp$ which satisfies $A\vec{X}_{R[A^*]}$ = $\vec{b}_{R[A]}$. By doing this, we will minimize (6-43) and $\left\|\vec{X}\right\|_2$ simultaneously. However, this optimum $\vec{X}$ is given by

$$\vec{X}_{R[A^*]}=A^+\vec{b}_{R[A]}=A^+\left[\vec{b}_{R[A]}\oplus\vec{b}_{K[A^*]}\right]=A^+\vec{b}. \tag{6-44}$$

From this, we conclude that the pseudo inverse solves the general optimization problem introduced by the paragraph containing Equation (6-1). Finally, when you use $\vec{X}$ = $A^+\vec{b}$, the norm of the residual becomes $\left\|\vec{b}_{K[A^*]}\right\|_2$; folks, it just doesn't get any smaller than this!

Let's give a geometric interpretation of our result. Given $\vec{b}$ $\in$ **V**, the orthogonal projection of $\vec{b}$ on the range of A is $\vec{b}_{R[A]}$. Hence, when solving the general minimum norm, least-square error problem, we first project $\vec{b}$ on R(A) to obtain $\vec{b}_{R[A]}$. Then we solve $A\vec{X}$ = $\vec{b}_{R[A]}$ for its minimum norm solution, a "Problem #1"-type problem.

Folks, this stuff is worth repeating and illustrating with a graphic! Again, to solve the general problem outlined on page 6-1, we

1) Orthogonally project $\vec{b}$ onto R(A) to get $\vec{b}_{R[A]}$,

2) Solve $A\vec{X}$ = $\vec{b}_{R[A]}$ for $\vec{X}\in$ R(A*) = K(A)$^\perp$ (a "Problem #1" exercise).

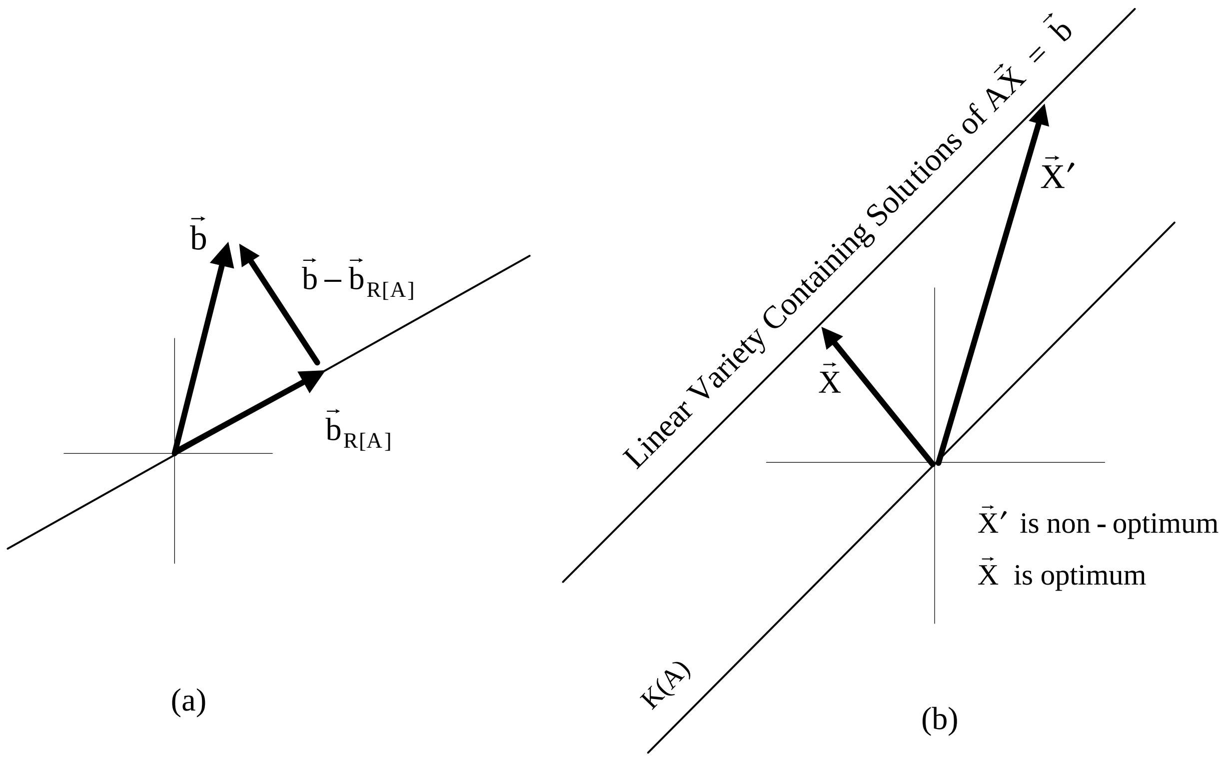


Figure 6-6: a) $\vec{b}_{R[A]}$ is the orthogonal projection of $\vec{\mathbf{b}}$ onto R(A). b) Find the minimum norm solution to $A\vec{\mathbf{X}} = \vec{b}_{R[A]}$ (this is a "Problem #1"-type problem).

This two-step procedure produces an optimum $\vec{X}$ that minimizes $\left\|A\vec{X} - \vec{b}\right\|_2$ *and* $\left\|\vec{X}\right\|_2$ simultaneously! It is illustrated by Fig. 6-6.

Application of the Pseudo Inverse: Least Squares Curve Fitting

We want to pass a straight line through a set of n data points. We want to do this in such a manner that minimizes the squared error between the line and the data. Suppose we have the data points (x_k, y_k), $1 \le k \le n$. As illustrated by Figure 6-7, we desire to fit the straight line

$$y = a_0 x + a_1 \tag{6-45}$$

to the data. We want to find the coefficients a_0 and a_1 that minimize

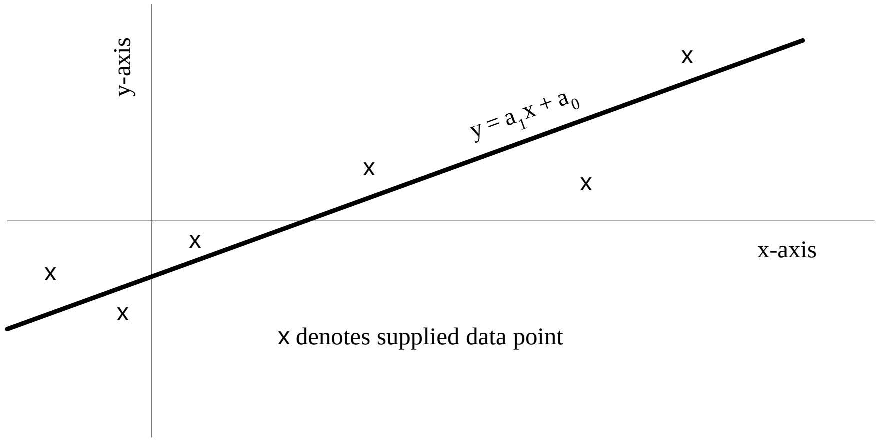


Figure 6-7: Least-squares fit a line to a data set.

$$\text{Error}^2 = \sum_{k=1}^{n}[y_k - (a_1x_k + a_0)]^2 . \tag{6-46}$$

This problem can be formulated as an overdetermined linear system, and it can be solved by using the pseudo inverse operator. On the straight line, define $\tilde{y}_k$, $1 \le k \le n$, to be the ordinate values that correspond to the x_k, $1 \le k \le n$. That is, define $\tilde{y}_k = a_1x_k + a_0$, $1 \le k \le n$, as the line points. Now, we adopt the vector notation

$$\begin{aligned}\vec{Y}_L &\equiv \begin{bmatrix}\tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n\end{bmatrix}^T \\ \vec{Y}_d &\equiv \begin{bmatrix}y_1 & y_2 & \cdots & y_n\end{bmatrix}^T\end{aligned}, \tag{6-47}$$

where $\vec{Y}_L$ denotes "line values", and $\vec{Y}_d$ denotes "y-data". Now, denote the "x-data" n×2 matrix as

$$X_d = \begin{bmatrix}1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n\end{bmatrix}. \tag{6-48}$$

Finally, the "line equation" is

$$X_d\begin{bmatrix}a_0\\a_1\end{bmatrix}=\vec{Y}_L\,, \tag{6-49}$$

which defines a straight line with slope a_1. Our goal is to select $[a_0 \;\; a_1]^T$ to minimize

$$\left\|\vec{Y}_L-\vec{Y}_d\right\|_2^2=\left\|\vec{X}_d\begin{bmatrix}a_0\\a_1\end{bmatrix}-\vec{Y}_d\right\|_2^2. \tag{6-50}$$

Unless all of the data lay on the line, the system

$$\vec{X}_d\begin{bmatrix}a_0\\a_1\end{bmatrix}=\vec{Y}_d \tag{6-51}$$

is inconsistent, so we want a "least squares fit" to minimize (6-50). From our previous work, we know the answer is

$$\begin{bmatrix}a_0\\a_1\end{bmatrix}=X_d^+\,\vec{Y}_d\,, \tag{6-52}$$

where X_d^+ is the pseudo inverse of X_d. Using (6-22), we write

$$\begin{bmatrix}a_0\\a_1\end{bmatrix}=\left(X_d^T X_d\right)^{-1}X_d^T\vec{Y}_d\,, \tag{6-53}$$

X_d is n×2 with rank 2. Hence $X_d^T X_d$ is a 2×2 positive definite symmetric (and nonsingular) matrix.

Example

Consider the 5 points

k	x_k	y_k
1	0	0
2	1	1.4
3	2	2.2
4	3	3.5
5	5	4.4

$$X_d = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}, \quad \vec{Y}_d = \begin{bmatrix} 0 \\ 1.4 \\ 2.2 \\ 3.5 \\ 4.4 \end{bmatrix}$$

MatLab yields the pseudo inverse

$$X_d^+ = \begin{bmatrix} .5270 & .3784 & .2297 & .0811 & -.2162 \\ -.1486 & -.0811 & -.0135 & .0541 & .1892 \end{bmatrix}$$

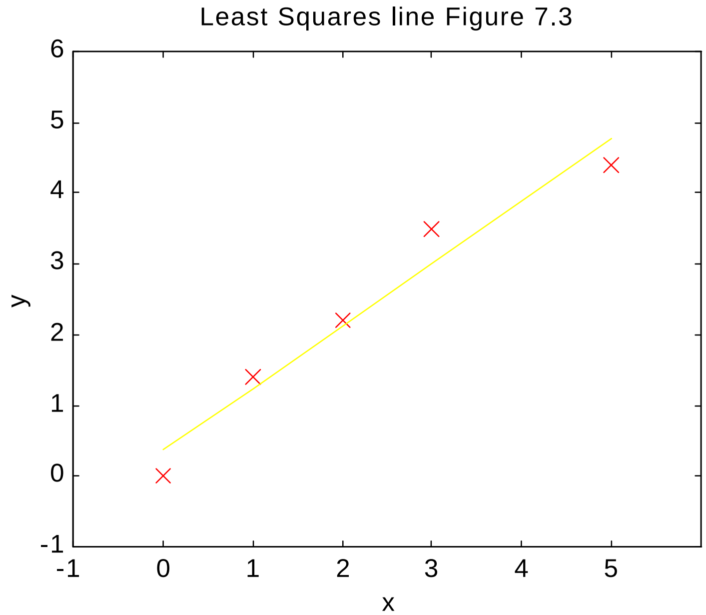
so that the coefficients a_0 and a_1 are

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = X_d^+ \vec{Y}_d = \begin{bmatrix} .3676 \\ .8784 \end{bmatrix}$$

The following MatLab program plots the points and the straight line approximation.

```
% EX7_3.M Least-squares curve fit with a line using
%  \ operator and polyfit. The results are displayed
%  and plotted.
x=[0 1 2 3 5];           % Define the data points
y=[0 1.4 2.2 3.5 4.4];
A1=[1 1 1 1 1 ]';        % Least squares matrix
A=[A1 x'];
Als=A'*A;
```

```
bls=A'*y';
% Compute least squares fit
Xlsq1=Als\bls;
Xlsq2=polyfit(x,y,1);
f1=polyval(Xlsq2,x);
error=y-f1;
disp('          x              y
f1        y-f1')
table=[x' y' f1' error'];
disp(table)
fprintf('Strike a key for the
plot\n')
pause
% Plot
clf
plot(x,y,'xr',x,f1,'-')
axis([-1 6 -1 6])
title('Least    Squares    line
Figure 7.3')
xlabel('x')
ylabel('y')
```



The output of the program follows.

x	y	f1	y-f1
0	0	0.3676	-0.3676
1.0000	1.4000	1.2459	0.1541
2.0000	2.2000	2.1243	0.0757
3.0000	3.5000	3.0027	0.4973
5.0000	4.4000	4.7595	-0.3595

Strike a key for the plot

N-Port Resistive Networks: Application of Pseudo Inverse

Consider the n-port electrical network. We treat the n-port as a *black box* that is

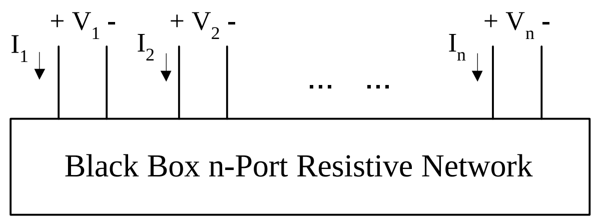


Figure 6-8: n-port network described by terminal behavior.

characterized by its terminal behavior. Suppose there are only *resistors and linearly dependent sources* in the black box. Then we can characterize the network using complex-valued voltages and currents (*i.e.*, "phasors") by a set of equations of the form

$$\begin{aligned}
V_1 &= z_{11}I_1 + z_{12}I_2 + \cdots + z_{1n}I_n \\
V_2 &= z_{21}I_1 + z_{22}I_2 + \cdots + z_{2n}I_n \\
\vdots & \quad\;\; \vdots \qquad\;\; \vdots \qquad\qquad\;\; \vdots \\
V_n &= z_{n1}I_1 + z_{n2}I_2 + \cdots + z_{nn}I_n
\end{aligned} \tag{6-54}$$

This is equivalent to writing $\vec{V} = Z\vec{I}$ where $\vec{V} = [V_1 \;\; V_2 \;\; \ldots \;\; V_n]^T$, $\vec{I} = [I_1 \;\; I_2 \;\; \ldots \;\; I_n]^T$ and

$$Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} \tag{6-55}$$

is the n×n *impedance matrix*. The z_{ik}, $1 \le i,k \le n$, are known as *open circuited impedance parameters* (they are real-valued since we are dealing with resistive networks).

The network is said to be *reciprocal* if input port i can be interchanged with output port j and the relationship between port voltages and currents remain unchanged. That is, the network is reciprocal if Z is Hermitian ($Z^* = Z^T$ since the matrix is real-valued). Also, for a reciprocal resistive n-port, it is possible to show that Hermitian Z is positive semi-definite.

Suppose we connect two reciprocal n-ports in parallel to form a new reciprocal n-port. Given impedance matrices Z_1 and Z_2 for the two n-ports, we want to find the impedance matrix Z_{equ} of the parallel combination. Before doing this, we must give some preliminary material.

Theorem 6-6

Given n×n impedance matrices Z_1 and Z_2 of two resistive, reciprocal n port networks. Then

$$R(Z_1) + R(Z_2) = R(Z_1 + Z_2)\,. \tag{6-56}$$

Proof: As discussed above, we know that Z_1 and Z_2 are Hermitian and non-negative definite. We prove this theorem by showing

1) $R(Z_1 + Z_2) \subset R(Z_1) + R(Z_2)$, and

2) $R(Z_1) \subset R(Z_1 + Z_2)$ and $R(Z_2) \subset R(Z_1 + Z_2)$ so that $R(Z_1) + R(Z_2) \subset R(Z_1 + Z_2)$.

We show #1. Consider $\vec{Y} \in R(Z_1 + Z_2)$. Then there exists an $\vec{X}$ such that $\vec{Y} = (Z_1 + Z_2)\vec{X} = Z_1\vec{X} + Z_2\vec{X}$. But $Z_1\vec{X} \in R(Z_1)$ and $Z_2\vec{X} \in R(Z_2)$. Hence, $\vec{Y} \in R(Z_1) + R(Z_2)$, and this implies that $R(Z_1 + Z_2) \subset R(Z_1) + R(Z_2)$.

We show #2. Let $\vec{Y} \in K(Z_1+Z_2)$ so that $(Z_1+Z_2)\vec{Y} = \vec{0}$. Then

$$0 = \left\langle \vec{Y}, (Z_1 + Z_2)\vec{Y} \right\rangle = \left\langle (Z_1 + Z_2)\vec{Y}, \vec{Y} \right\rangle = \left\langle Z_1\vec{Y}, \vec{Y} \right\rangle + \left\langle Z_2\vec{Y}, \vec{Y} \right\rangle . \tag{6-57}$$

Now, Z_1 and Z_2 are non-negative definite so that both inner products on the right-hand side of (6-57) must be non-negative. Hence, we must have

$$\left\langle Z_1\vec{Y}, \vec{Y} \right\rangle = \left\langle Z_2\vec{Y}, \vec{Y} \right\rangle = 0\,. \tag{6-58}$$

Since Z_1 and Z_2 are Hermitian, Equation (6-58) can be true if and only if $Z_1\vec{Y} = \vec{0}$ and $Z_2\vec{Y} = \vec{0}$; that is, $\vec{Y} \in K(Z_1)$ and $\vec{Y} \in K(Z_2)$. Hence, we have shown that

$$\begin{aligned} &K(Z_1 + Z_2) \subset K(Z_1),\;\; K(Z_1 + Z_2) \subset K(Z_2) \\ &R(Z_1) \subset R(Z_1 + Z_2),\; R(Z_2) \subset R(Z_1 + Z_2) \end{aligned} \tag{6-59}$$

Finally, the second set relationship in (6-59) implies $R(Z_1) + R(Z_2) \subset R(Z_1+Z_2)$, and the theorem is proved.♥

Parallel Sum of Matrices

Let N_1 and N_2 be reciprocal n-port resistive networks that are described by n×n matrices Z_1 and Z_2, respectively. The *parallel sum* of Z_1 and Z_2 is denoted as $Z_1 : Z_2$, and it is *defined* as

$$Z_1 : Z_2 = Z_1 (Z_1 + Z_2)^+ Z_2 , \tag{6-60}$$

an n×n matrix.

Theorem 6-7

Let Z_1 and Z_2 be impedance matrices of reciprocal n-port resistive networks. Then, we have

$$Z_1 : Z_2 = Z_1 (Z_1 + Z_2)^+ Z_2 = Z_2 (Z_1 + Z_2)^+ Z_1 = Z_2 : Z_1 \tag{6-61}$$

Proof:

By inspection, the matrix $(Z_1 + Z_2)(Z_1 + Z_2)^+(Z_1 + Z_2)$ is symmetric. Multiply out this matrix to obtain

$$\begin{aligned}&(Z_1 + Z_2)(Z_1 + Z_2)^+(Z_1 + Z_2)\\ &\qquad = [Z_1(Z_1 + Z_2)^+Z_1 + Z_2(Z_1 + Z_2)^+Z_2] + [Z_1(Z_1 + Z_2)^+Z_2 + Z_2(Z_1 + Z_2)^+Z_1] .\end{aligned} \tag{6-62}$$

The left-hand-side of this result is symmetric. Hence, the right-hand-side must be symmetric as well. This requires that both $Z_1(Z_1 + Z_2)^+Z_2$ and $Z_2(Z_1 + Z_2)^+Z_1$ be symmetric, so that

$$\begin{aligned}Z_1\left(Z_1 + Z_2\right)^+ Z_2 &= \left(Z_1(Z_1 + Z_2)^+ Z_2\right)^T = Z_2^T\left((Z_1 + Z_2)^+\right)^T Z_1^T = Z_2^T\left((Z_1 + Z_2)^T\right)^+ Z_1^T\\ &= Z_2\left(Z_1 + Z_2\right)^+ Z_1\end{aligned} \tag{6-63}$$

as claimed. Hence, we have $Z_1 : Z_2 = Z_2 : Z_1$ and the parallel sum of symmetric matrices is a symmetric matrix.

Theorem 6-8: Parallel n-Port Resistive Networks

Suppose that N_1 and N_2 are two reciprocal n-port resistive networks that are described by impedance matrix Z_1 and Z_2, respectively. Then the parallel connection of N_1 and N_2 is described by the impedance matrix

$$Z_{equ} = Z_1 : Z_2 = Z_1(Z_1 + Z_2)^{+}Z_2\,, \tag{6-64}$$

the parallel sum of matrices Z_1 and Z_2.

Proof:

Let $\vec{I}_1$, $\vec{I}_2$ and $\vec{I} = \vec{I}_1 + \vec{I}_2$ be the current vectors entering networks N_1, N_2 and the parallel combination of N_1 and N_2, respectively. Likewise, let $\vec{V}$ be the voltage vector that is supplied to the parallel connection of N_1 and N_2. To prove this theorem, we must show that

$$\vec{V} = Z_1(Z_1 + Z_2)^{+}Z_2\vec{I} = (Z_1 : Z_2)\vec{I}. \tag{6-65}$$

Since $\vec{V}$ is supplied to the parallel combination, we have

$$\vec{V} = Z_1\vec{I}_1 = Z_2\vec{I}_2\,. \tag{6-66}$$

Use this last equation to write

$$\begin{aligned} Z_1\vec{I} &= Z_1(\vec{I}_1 + \vec{I}_2) = \vec{V} + Z_1\vec{I}_2 \\ Z_2\vec{I} &= Z_2(\vec{I}_1 + \vec{I}_2) = \vec{V} + Z_2\vec{I}_1 \end{aligned}\,. \tag{6-67}$$

Now, multiply these last two equations by $(Z_1 + Z_2)^{+}$ to obtain

$$
\begin{aligned}
(Z_1 + Z_2)^+ Z_1\vec{I} &= (Z_1 + Z_2)^+ \vec{V} + (Z_1 + Z_2)^+ Z_1\vec{I}_2 \\
(Z_1 + Z_2)^+ Z_2\vec{I} &= (Z_1 + Z_2)^+ \vec{V} + (Z_1 + Z_2)^+ Z_2\vec{I}_1
\end{aligned}
\quad . \qquad (6\text{-}68)
$$

On the left, multiply the first of these by Z_2 and the second by Z_1 to obtain (using the parallel sum notation)

$$
\begin{aligned}
(Z_2{:}Z_1)\vec{I} &= Z_2(Z_1 + Z_2)^+ \vec{V} + (Z_2{:}Z_1)\vec{I}_2 \\
(Z_1{:}Z_2)\vec{I} &= Z_1(Z_1 + Z_2)^+ \vec{V} + (Z_1{:}Z_2)\vec{I}_1
\end{aligned}
\quad . \qquad (6\text{-}69)
$$

However, we know that $Z_1{:}Z_2 = Z_2{:}Z_1$, $\vec{I} = \vec{I}_1 + \vec{I}_2$, and Z_1+Z_2 is symmetric. Hence, when we add the two equation (6-69) we get

$$
\begin{aligned}
(Z_1{:}Z_2)\vec{I} &= (Z_1 + Z_2)(Z_1 + Z_2)^+ \vec{V} \\
&= P\vec{V}
\end{aligned}
\quad , \qquad (6\text{-}70)
$$

where

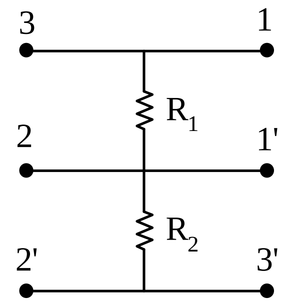
$$P \equiv (Z_1 + Z_2)(Z_1 + Z_2)^+ \qquad (6\text{-}71)$$

is an orthogonal projection operator onto the range of $Z_1 + Z_2$. Since $\vec{V} = Z_1\vec{I}_1 = Z_2\vec{I}_2$, we know that $\vec{V} \in R(Z_1)$ and $\vec{V} \in R(Z_2)$. However, by Theorem 6-6, we know that $R(Z_1) \subset R(Z_1 + Z_2)$ and $R(Z_2) \subset R(Z_1 + Z_2)$. Hence, we have $\vec{V} \in R(Z_1 + Z_2)$ so that $P\vec{V} = \vec{V}$, where P is the orthogonal projection operator (6-71). Hence, from (6-70), we conclude that

$$(Z_1{:}Z_2)\vec{I} = P\vec{V} = \vec{V}, \qquad (6\text{-}72)$$

so the parallel sum $Z_1{:}Z_2$ is the impedance matrix for the parallel combination of N_1 and N_2.♥

Example



Consider the 3-port networks shown to the right (1 to 1', 2 to 2' and 3 to 3' are the ports; voltages are sensed positive, and currents are defined as entering, the unprimed terminals 1, 2 and 3). For $1 \le i, j \le 3$, the entries in the impedance matrix are

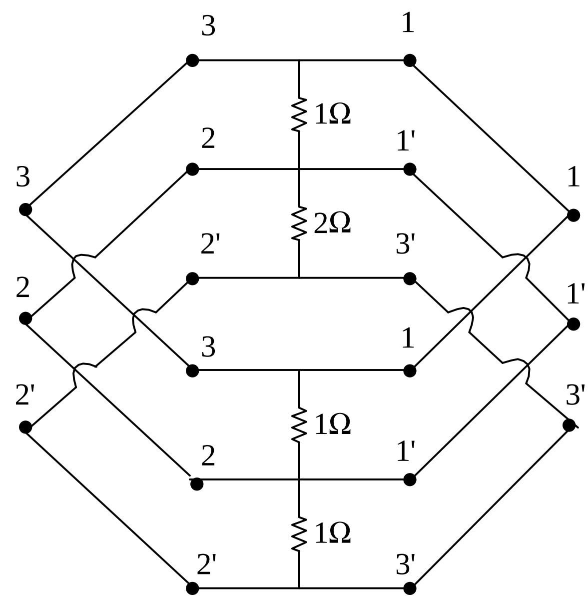
$$z_{ij} \equiv \frac{V_i}{I_j},$$

where V_i is the voltage across the i^{th} port (voltage is signed positive at the unprimed port terminal), and I_j is the current entering the j^{th} port (current defined as entering unprimed port terminal). Elementary circuit theory can be used to obtain

$$Z = \begin{bmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ R_1 & R_2 & R_1 + R_2 \end{bmatrix}$$

as the impedance matrix for this 3-port.

Connect two of these 3-port networks in parallel to form the network depicted to the right. The impedance matrices are



$$Z_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

both of which are singular. By Theorem 6-8, the impedance matrix for the parallel combination is

$$Z_{equ} = Z_1 \colon Z_2 = Z_1 (Z_1 + Z_2)^+ Z_2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \right)^+ \left[\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \right] = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 2/3 & 2/3 \\ 1/2 & 2/3 & 7/6 \end{bmatrix} .$$