ECE 307: Electricity and Magnetism
Fall 2012

Instructor:  J.D. Williams, Assistant Professor
Electrical and Computer Engineering
University of Alabama in Huntsville
406 Optics Building, Huntsville, Al 35899
Phone: (256) 824-2898, email: john.williams@uah.edu
Course material posted on UAH Angel course management website

Textbook:

Optional Reading:

All figures taken from primary textbook unless otherwise cited.
Chapter 8: Magnetic Forces Materials and Devices

• Topics Covered
  – Forces Due to Magnetic Fields
  – Magnetic Torque and Moment
  – A Magnetic Dipole
  – Magnetization in Materials
  – Classification of Magnetic Materials
  – Magnetic Boundary Conditions
  – Inductors and Inductances
  – Magnetic Energy
  – Magnetic Circuits
  – Forces on Magnetic Materials

• Homework: 2, 5, 16, 24, 25, 32, 33, 38, 39, 40, 41, 46, 52, 53

All figures taken from primary textbook unless otherwise cited.
Lorentz Force Law

- The force on a charged particle in an electric field is simply \( F = qE \).
- However, in the presence of an electromagnetic field an additional force is imposed from the charge displacement of velocity, \( \mathbf{u} \), quantified by the magnetic field, \( \mathbf{B} \).
- The combined force is defined by Lorentz Force Law: \( \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \).
- Equating the Lorentz force to Newton’s force equation, one obtains \( \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = m\mathbf{a} = m\frac{d\mathbf{u}}{dt} \) where \( \mathbf{a} \) is the acceleration of the particle.
- Requiring that the kinetic energy of a charged particle in an electric field is therefore
  \[
  \mathbf{F} = q(\mathbf{E}) = m\frac{d\mathbf{u}}{dt}
  \]
  
  \[
  u_x = \int \frac{qE_x}{m} dt
  \]
  
  \[
  u_y = \int \frac{qE_y}{m} dt
  \]
  
  \[
  u_z = \int \frac{qE_z}{m} dt
  \]

  \[
  KE = \frac{1}{2} m|\mathbf{u}|^2
  \]

The location of the particle can also be found as

\[
\mathbf{u} = \frac{d\mathbf{l}}{dt}
\]

\[
l_i = \int u_i dt
\]

8/17/2012
Forces Due to a Magnetic Field

- Similarly, one can solve for the kinetic energy of a particle in a magnetostatic field

\[ \vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) = m\vec{a} = m \frac{d\vec{u}}{dt} \]

- The kinetic energy of a charged particle in an magnetic field is therefore

\[ KE = \frac{1}{2} m|\vec{u}|^2 \]

For \( \vec{B} \), \( \vec{u} \), and \( \vec{a} \) in orthogonal directions, one can deduce a coordinate system in which

\[ u_1 = \int \frac{q(\vec{v} \times \vec{B})}{m} dt = \int \frac{q|\vec{v}|\vec{B}}{m} dt = \int \omega|\vec{v}| dt \]

\[ u_2 = \int \frac{q(\vec{v} \times \vec{B})}{m} dt = |u_1|\hat{u}_2 \]

\[ \omega = \frac{q|\vec{B}|}{m} \]

\[ u_3 = \int \frac{q(\vec{v} \times \vec{B})}{m} dt = 0 \]

Cyclotron Resonance Frequency

The location of the particle can also be found as

\[ \vec{u} = \frac{d\vec{l}}{dt} \]

\[ l_i = \int u_i dt \]
Velocity of a Particle with no External Forces

- Charged particles traveling at constant velocities within an electromagnetic field have no acceleration term requiring that the total force due to the electric and magnetic field equal zero. In this case, the velocity is the ratio of the E and B fields

\[ \vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) = 0 \]

\[ qE = quB \]

\[ u = \frac{E}{B} \]

- Critical speed required to balance the two parts of the Lorentz force
- Particles traveling at this speed travel along a strait path
- Particles traveling slower than this speed have travel in arc toward the magnetic force
- Particles traveling faster than this speed travel in an arc toward the electric force
- Thus one can design a particle filter
Force on a Current Element

• One can also use field calculations to determine the force acting on a current element, \( I dl = \mathbf{K} dS = J dv \), due to an applied external magnetic field \( \mathbf{B} \).

• Assume that a copper wire carries a current density, \( \mathbf{J} = \rho_v \mathbf{u} \).

• We know:
  \[ I dl = J dv = \rho_v \mathbf{u} = dQ \mathbf{u} \]

• And that:
  \[ \vec{F} = q(\vec{E} + \mathbf{u} \times \vec{B}) \]
  where \( \vec{E} \rightarrow 0 \)

  \[ \vec{F} = q(\mathbf{u} \times \vec{B}) \]

Thus we can solve for the force on the first wire:

\[ d\vec{F} = dq(\mathbf{u} \times \vec{B}) = I dl \times \vec{B} \]

\[ \vec{F} = \int_L I dl \times \vec{B} \]

Likewise

\[ \vec{F} = \int_L K dS \times \vec{B} = \vec{F} = \int_L J dv \times \vec{B} \]

• Requiring one to define the magnetic field, \( \mathbf{B} \), as the force per unit current element
Force Between 2 Current Elements

- Now one can use defined the force on a current element from a magnetic field.
- However, that magnetic field must have been generated somehow. What if it was generated by field produced from current passing through a second current element nearby. This means that currents in neighboring wires generate magnetic fields that generate forces on each other.
- Newton’s law requires that the force, $\mathbf{F}_1$, acting on element 1 is equal and opposite to the force $\mathbf{F}_2$ acting on element 2.
- One can calculate these interdependent forces through the following derivation.

\[
d\vec{F}_1 = I_1 d\vec{l}_1 \times \vec{B}_2 \\
d(d\vec{F}_1) = I_1 d\vec{l}_1 \times d\vec{B}_2 \\
\text{recall Biot Savart's Law} \\
d\vec{B}_2 = \frac{\mu_0 I_2 d\vec{l}_2 \times \hat{a}_R}{4\pi R_{12}^2}
\]

Substitution

\[
d(d\vec{F}_1) = I_1 d\vec{l}_1 \times \frac{\mu_0 (I_2 d\vec{l}_2 \times \hat{a}_R)}{4\pi R_{12}^2} \\
d\bar{F}_1 = \int \int_{L_1} I_1 d\vec{l}_1 \times \frac{\mu_0 (I_2 d\vec{l}_2 \times \hat{a}_R)}{4\pi R_{12}^2} \\
\bar{F}_1 = \int \int_{L_1} I_1 d\vec{l}_1 \times \frac{\mu_0 (I_2 d\vec{l}_2 \times \hat{a}_R)}{4\pi R_{12}^2} = \mu_0 I_1 I_2 \int \int_{L_1} d\vec{l}_1 \times \frac{(d\vec{l}_2 \times \hat{a}_R)}{R_{12}^2}
\]

8/17/2012
Consider a rectangular loop of current \( I_1 = I_l \) placed near an infinitely long wire filament carrying current \( I_2 = I_w \).

Let’s examine the force on the sections of the loop parallel to the line charge.
Consider a rectangular loop of current $I_1 = I_l$ placed near an infinitely long wire filament carrying current $I_2 = I_w$.

Let’s examine the force on the sections of the loop perpendicular to the line charge.

$$\vec{F}_1 = \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \int I_1 d\vec{l}_1 \times \vec{B}_w$$

$$\vec{F}_1 + \vec{F}_3 = -\frac{\mu_0 I_1 I_2 b}{2\pi \rho_0} \hat{\rho} + \frac{\mu_0 I_1 I_2 b}{2\pi (\rho_0 + a)} \hat{\rho} = -\frac{\mu_0 I_1 I_2 b}{2\pi} \frac{1}{\rho} \left( \frac{1}{\rho_0} - \frac{1}{\rho_0 + a} \right) \hat{\rho}$$

$$\vec{F}_2 + \vec{F}_4 = 0$$

For $\vec{F}_2$

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi \rho} \hat{\phi}$$

$$d\vec{l}_2 = d\rho \hat{\rho}$$

$$\vec{F}_2 = \int I_2 d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi \rho} \hat{\phi} = \frac{\mu_0 I_1 I_2}{2\pi} \int_{\rho = \rho_0}^{\rho_0 + a} d\rho \hat{\rho} \times \frac{1}{\rho} \hat{\phi} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \left( \frac{\rho_0 + a}{\rho_0} \right) \hat{z}$$

For $\vec{F}_2$

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi \rho} \hat{\phi}$$

$$d\vec{l}_2 = d\rho \hat{\rho}$$

$$\vec{F}_2 = \int I_2 d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi \rho} \hat{\phi} = \frac{\mu_0 I_1 I_2}{2\pi} \int_{\rho = \rho_0}^{\rho_0 + a} d\rho \hat{\rho} \times \frac{1}{\rho} \hat{\phi} = \frac{\mu_0 I_1 I_2}{2\pi} \int_{\rho = \rho_0}^{\rho_0 + a} d\rho \hat{\phi} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \left( \frac{\rho_0 + a}{\rho_0} \right) \hat{z}$$
Magnetic Torque and Moment
(show class demo here)

• Now that we have examined the force on a current carrying loop. Let's examine the Torque applied to it.
• Torque, \( \mathbf{T} \), on the loop is the vector product of the force, \( \mathbf{F} \), and the moment arm, \( \mathbf{r} \).

\[
\mathbf{T} = \mathbf{r} \times \mathbf{F}
\]

\[
|\mathbf{T}| = |\mathbf{r}| |\mathbf{F}| \sin \alpha
\]

\[
\text{and \ for \ uniform \ } \mathbf{B}
\]

\[
|\mathbf{F}_0| = IBl
\]

\[
|\mathbf{T}| = IBlw \sin \alpha
\]

but

\[
lw = S
\]

yielding

\[
|\mathbf{T}| = IBS \sin \alpha
\]

Where we can now define a quantity \( \mathbf{m} \) as the magnetic dipole moment with units \( \text{A/m}^2 \) which is the product of the current and area of the loop in the direction normal the surface area defined by the loop

\[
\mathbf{m} = IS\mathbf{\hat{n}}
\]

\[
\mathbf{T} = \mathbf{m} \times \mathbf{B}
\]
Magnetic Dipole

- A bar magnet or small filament loop is generally referred to as a magnetic dipole
- Let us determine the magnetic field, \( B \), of a dipole from an observation point \( P(r, \theta, \phi) \) due to a circular loop carrying current, \( I \).
- Torque, \( T \), on the loop is the vector product of the force, \( F \), and the moment arm, \( r \).

\[
\vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}}{r} = \vec{A} = \frac{\mu_0 I\pi a^2}{4\pi r^2} \hat{a}_\phi = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{a}_r
\]

where
\[
\vec{m} = I\pi a^2 \hat{a}_z
\]
\[
\hat{a}_z \times \hat{a}_r = (\sin \theta) \hat{a}_\phi
\]

\[
\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 |\vec{m}|}{4\pi r^3} (2\cos \theta \hat{a}_r + (\sin \theta) \hat{a}_\theta)
\]

Note the comparison between the magnetic dipole term and that developed for the electrostatic dipole in Chapter 4.

\[
V = \frac{\vec{p} \cdot \hat{a}_r}{4\pi \varepsilon_0 r^2} = \frac{q}{4\pi \varepsilon_0} \left( \frac{d \cos \theta}{r^2} \right)
\]

\[
\vec{E} = -\nabla V = \frac{|\vec{p}|}{4\pi \varepsilon_0 r^3} (2\cos \theta \hat{a}_r + (\sin \theta) \hat{a}_\theta)
\]
Comparison of Electric and Magnetic Dipoles

<table>
<thead>
<tr>
<th>Electric</th>
<th>Magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ V = \frac{Q}{4\pi\varepsilon_0 r} ]</td>
<td>Does not exist</td>
</tr>
<tr>
<td>[ E = \frac{Qa_r}{4\pi\varepsilon_0 r^2} ]</td>
<td></td>
</tr>
<tr>
<td>Monopole (point charge)</td>
<td>Monopole (point charge)</td>
</tr>
<tr>
<td>[ V = \frac{Q \cos \theta}{4\pi\varepsilon_0 r^2} ]</td>
<td>[ A = \frac{\mu_0 m \sin \theta a_\phi}{4\pi r^2} ]</td>
</tr>
<tr>
<td>[ E = \frac{Qd}{4\pi\varepsilon_0 r^3} (2 \cos \theta a_r + \sin \theta a_\theta) ]</td>
<td>[ B = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta a_r + \sin \theta a_\theta) ]</td>
</tr>
<tr>
<td>Dipole (two-point charge)</td>
<td>Dipole (small current loop or bar magnet)</td>
</tr>
</tbody>
</table>
Torque and Dipole Properties of a Bar Magnet

- A bar magnet or small filament loop is generally referred to as a magnetic dipole.
- Assume a bar magnetic of length, l, generates a uniform magnetic field, B, and a dipole moment, |m|=Qm|
- Torque, T, on the loop is the vector product of the force, F, and the moment arm, r.

Thus, because the bar magnet represents a magnetic dipole moment equal in magnitude to the dipole moment of a current loop, a bar magnet can also be taken as a magnetic dipole.

\[
\vec{T} = \vec{m} \times \vec{B} = Q_m \vec{l} \times \vec{B}
\]
\[
\vec{F} = Q_m \vec{B}
\]
\[
|\vec{T}| = Q_m lB = ISB
\]
\[
\Rightarrow Q_m l = IS
\]

Therefore the field at a reasonable distance away from any bar magnet is mathematically identical to that of a dipole.
Dipole Moment (Example)

- Consider a small current loop $L_1$ with a magnetic moment $m_1 = 5 \mathbf{a}_z \text{ Am}^2$ located at the origin.
- Another loop $L_2$ with magnetic moment $m_2 = 3 \mathbf{a}_y \text{ Am}^2$ is located at $P (4, -2, 10)$.
- What is the torque on $L_2$?

$$\vec{T}_2 = \vec{m}_2 \times \vec{B}_1$$

$$\vec{B}_1 = \frac{\mu_0 |m_1|}{4\pi r^3} (2(\cos \theta)\hat{a}_r + (\sin \theta)\hat{a}_\theta)$$

$$\vec{m}_2 = 3\hat{a}_y = 3((\sin \theta)(\cos \phi)\hat{a}_r + (\cos \theta)(\sin \phi)\hat{a}_\theta + (\cos \theta)\hat{a}_\phi)$$

$$|\vec{r}| = |\vec{r}_P| = \sqrt{4^2 + (-3)^2 + 10^2} = 5\sqrt{5}$$

$$\tan \theta = \frac{\rho}{z} = \frac{5}{10}$$

$$\sin \theta = \frac{1}{\sqrt{5}}, \cos \theta = \frac{2}{\sqrt{5}}$$

$$\tan \phi = \frac{y}{x} = -\frac{3}{4}, \sin \phi = -\frac{3}{5}; \cos \phi = \frac{4}{5}$$

$$|\vec{m}_1| = 5$$

$$\vec{m}_2 = 3\left(-\frac{3\hat{a}_r}{5\sqrt{5}} - \frac{6\hat{a}_\theta}{5\sqrt{5}} + \frac{4\hat{a}_\phi}{5}\right)$$

$$\vec{T}_2 = \frac{3\mu_0}{4\pi(3125)\sqrt{5}} \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ -3 & -6 & 4\sqrt{5} \\ 4 & 1 & 0 \end{vmatrix}$$

8/17/2012
Magnetization in Materials

- We know that all materials are made up of atoms consisting of electrons orbiting nuclei.
- Each of these electrons can also be said to spin about its axis.
- In certain materials these spins associated with atomic magnetic dipoles align over large atomic distances to create magnetic domains across several thousands of atoms.
- As the individual magnetic domains align, over larger and larger volumes of the material, then the material is said to magnetize.
- Magnetization $\mathbf{M}$, in A/m, is the magnetic dipole moment per unit volume.
- If $N$ atoms are in a given volume, $\Delta v$, then the $k^{th}$ atom has a magnetic moment $\mathbf{m}_k$.

\[ \mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{k=1}^{N} \mathbf{m}_k \]

\[ d\mathbf{m} = \mathbf{M} \, dv' \]

\[ d\mathbf{A} = \frac{\mu_0}{4\pi R^2} \left( \mathbf{M} \times \mathbf{a}_R \right) dv' \]

repeating

\[ \frac{\vec{R}}{R^3} = \nabla' \frac{1}{R} \]

then

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \left( \mathbf{M} \times \nabla' \frac{1}{R} \right) dv' \]
Magnetization in Materials (2)

- Substitution of the proper vector identities, coupled with the physics discussed in Chapter 7 will provide sufficient analysis to develop a complete working theory between magnetization, \( \mathbf{M} \), and the generation of \( \mathbf{A} \), \( \mathbf{J} \), \( \mathbf{B} \), and \( \mathbf{H} \).

\[
\vec{A} = \frac{\mu_0}{4\pi} \int \left( \mathbf{M} \times \nabla' \frac{1}{R} \right) dv'
\]

*recall* *the* *Identity*

\[
\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R}
\]

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_v \nabla' \times \frac{\mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_v \nabla' \times \frac{\mathbf{M}}{R} dv'
\]

*apply* *the* *identity*

\[
\int_v \nabla' \times \vec{F} dv' = -\oint_s \vec{F} \times d\mathbf{S}
\]

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_v \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \oint_s \frac{\mathbf{M} \times d\mathbf{S}}{R} = \frac{\mu_0}{4\pi} \int_v \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \oint_s \frac{\mathbf{M} \times \hat{a}_n}{R} dS
\]

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_v \frac{\mathbf{J}_b}{R} dv' + \frac{\mu_0}{4\pi} \oint_s \frac{\mathbf{K}_b dS'}{R}
\]

Where \( b \) in the \( \mathbf{J} \) and \( \mathbf{K} \) terms represents a bound current densities
Magnetization in Materials (3)

- Substitution of the proper vector identities, coupled with the physics discussed in Chapter 7 will provide sufficient analysis to develop a complete working theory between magnetization, \( \mathbf{M} \), and the generation of \( \mathbf{A} \), \( \mathbf{J} \), \( \mathbf{B} \), and \( \mathbf{H} \).

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}_b dv'}{R} + \frac{\mu_0}{4\pi} \oint_S \vec{K}_b dS'
\]

\[
\vec{J}_b = \nabla \times \vec{M}
\]

\[
\vec{K}_b = \vec{M} \times \hat{a}_n
\]

In _free _space, \( \Rightarrow \vec{M} = 0 \)

\[
\nabla \times \vec{H} = \vec{J}_f \iff \nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_f
\]

However, _in _a _material, \( \Rightarrow \vec{M} \neq 0 \)

\[
\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_f + \vec{J}_b = \vec{J} = \nabla \times \vec{H} + \nabla \times \vec{M}
\]

\( \Rightarrow \vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) \)

\[
\vec{M} = \chi_m \vec{H}
\]

\( \chi_m \) is called the magnetic susceptibility of the medium

\[
\vec{B} = \mu_0 \left( 1 + \chi_m \right) \vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H}
\]

_yielding_

\[
\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}
\]

Where \( \mu \) is called the permeability of the material and is measured in H/m

Henry (H) is the unit of inductance that will be defined later in this chapter

\( \mu_r \) is called the relative permeability

Where \( f \) in the \( \mathbf{J} \) term represents a free current density
In general we use the magnetic susceptibility (or relative permeability) to classify materials in terms of their magnetic property.

A material is said to be nonmagnetic if there is no bound current density or zero susceptibility. Otherwise it is magnetic.

Magnetic materials may be grouped into three classes: diamagnetic, paramagnetic, and ferromagnetic.

For many practical purposes, diamagnetic and paramagnetic materials exhibit little to no magnetic susceptibility. What magnetic properties these materials do have, follow a linear response over a large range of applied fields.

Ferromagnetic materials kept below the Curie temperature exhibit very large nonlinear magnetic susceptibility and are used for conventional magnetic device applications.
Classification of Magnetic Materials (2)

- **Diamagnetism**
  - Occurs when the magnetic fields in the material due to individual electron moments cancels each other out. Thus the permanent magnetic moment of each atom is zero.
  - Such materials are very weakly affected by magnetic fields.
  - Diamagnetic materials include Copper, Bismuth, silicon, diamond, and sodium chloride (table salt).
  - In general this effect is temperature independent. Thus, for example, there is no technique for magnetizing copper.
  - Superconductors exhibit perfect diamagnetism. The effect is so strong that magnetic fields applied across a superconductor do not penetrate more than a few atomic layers, resulting in $B=0$ within the material.

- **Paramagnetism**
  - Materials whose atoms exhibit a slight non-zero magnetic moment.
  - Paramagnetism is temperature dependent.
  - Most materials (air, tungsten, potassium, monell) exhibit paramagnetic effects that provide slight magnetization in the presence of large fields at low temperatures.

Ferromagnetism

- Occurs in atoms with a relatively large magnetic moment
- Examples: Cobalt, Iron, Nickel, various alloys based on these three
- Capable of being magnetized very strongly by a magnetic field
- Retain a considerable amount of their magnetization when removed from the field
- Lose their ferromagnetic properties and become linear paramagnetic materials (non magnetic) when the temperature is raised above a critical temperature called the Curie temperature.
- Their magnetization is nonlinear. Thus the constitutive relation $B=\mu_0\mu_r H$ does not hold because $\mu_r$ depends directly on $B$ and cannot be represented by a single value.

Ferromagnetic shielding

- Ferromagnetic materials can be used to “focus” and guide the flow of incident magnetic fields
- By placing a ferromagnetic material completely around a device, one can shield said device from an external field. This shielding occurs b/c the ferromagnet acts as a magnetic waveguide, that transmits the field around the shape of the structure and not within it.
Classification of Magnetic Materials (4)

- **Ferromagnetism - B-H Curve**
  - The magnetization of a ferromagnet in an external applied field, \(H\), is presented below.
  - As \(H\) is increased, the magnetic field, \(B\), within the material increases significantly and then begins to saturate to a value \(B_{\text{max}}\) as \(|H|\) approaches \(H_{\text{max}}\).
  - As the applied field, \(H\), is removed, the ferromagnetic material retains some degree of its magnetization until the point at which the applied field \(H\) is completely reversed at which time the magnetic field inside the material saturates to the \(-B_{\text{max}}\).
  - The applied field is then increased again to generate the complete Hysteresis curve.
  - Two other defining values are indicative of every B-H magnetization (Hysteresis) curve.
    - When the applied field is maxed and then again reduced to a zero value. The magnetic field within the material remains at some positive value \(B\), referred to as the permanent flux density.
    - The value upon which \(B\) becomes zero under an applied \(H\) value is called the coercive field intensity, \(H_c\).
    - Materials with small coercive field intensity values are said to be soft magnetic materials and do not retain significant magnetization upon the removal of the field.
    - Hard magnets (permanent magnets) have very large coercive field intensity values.
Magnetic Boundary Conditions (1)

- Magnetic boundary conditions for B and H crossing any material interface must match the following conditions developed using Guass’s law for magnetic fields and Ampere’s circuit law

\[ \oint B \cdot d\vec{S} = 0 \quad \oint H \cdot d\vec{l} = I \]

\[ \oint B \cdot d\vec{S} = 0 \]

\[ B_{1n} \Delta S - B_{2n} \Delta S = 0 \]

yields

\[ B_{1n} = B_{2n} \]

\[ \mu_1 H_{1n} = \mu_2 H_{2n} \]
Magnetic Boundary Conditions (2)

- Magnetic boundary conditions for B and H crossing any material interface must match the following conditions developed using Guass’s law for magnetic fields and Ampere’s circuit law

\[ \oint B \cdot d\bar{S} = 0 \quad \oint H \cdot d\bar{l} = I \]

\[ \oint H \cdot d\bar{l} = I \]

\[ K \Delta w = H_{1t} \Delta w + H_{1n} \frac{\Delta h}{2} + H_{2n} \frac{\Delta h}{2} - H_{2t} \Delta w - H_{2n} \frac{\Delta h}{2} - H_{1n} \frac{\Delta h}{2} \]

\[ \Delta h \to 0 \]

\[ K \Delta w = H_{1t} \Delta w - H_{2t} \Delta w \]

\[ H_{1t} - H_{2t} = K = \frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} \]

yields

\[ (\vec{H}_1 - \vec{H}_2) \times \hat{a}_{n12} = \vec{K} \]

\[ \vec{K} \to 0 \]

yields

\[ \vec{H}_{1t} = \vec{H}_{2t} \]

\[ \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2} \]
Magnetic Boundary Conditions (Review)

- Magnetic boundary conditions for B and H crossing any material interface must match the following conditions developed using Gauss's law for magnetic fields and Ampere’s circuit law:

\[ \oint B \cdot d\vec{S} = 0 \]

\[ \oint H \cdot d\vec{l} = I \]

\[ B_{1n} = B_{2n} \]

\[ \mu_1 H_{1n} = \mu_2 H_{2n} \]

\[ H_{1t} = H_{2t} \]

\[ B_{1t} = B_{2t} \]

\[ \frac{\mu_1}{\mu_2} \]
Magnetic Boundary Conditions (4)

- These boundary conditions can be used to develop an equivalent to Snell’s law for magnetic fields

\[ B_1 \cos \theta_1 = |\vec{B}_{1n}| = |\vec{B}_{2n}| = B_2 \cos \theta_2 \]
\[ \frac{B_1}{\mu_1} \sin \theta_1 = |\vec{H}_{1t}| = |\vec{H}_{2t}| = \frac{B_{2t}}{\mu_2} \sin \theta_2 \]
\[ \mu_2 \tan \theta_1 = \mu_1 \tan \theta_2 \]
Inductors and Inductance

• We now know that closed magnetic circuit carrying current I produces a magnetic field with flux

\[ \Psi = \int \vec{B} \cdot d\vec{S} \]

• We define the flux linkage between a circuit with N identical turns as

\[ \lambda = N \Psi \]

• As long as the medium the flux passes through is linear (isotropic) then flux linkage is proportional to the current I producing it and can be written as

\[ \lambda = LI \]

Where L is a constant of proportionality called the inductance of the circuit. A circuit that contains inductance is said to be an inductor.

• One can equate the inductance to the magnetic flux of the circuit as

\[ L = \frac{\lambda}{I} = \frac{N \Psi}{I} \]

where L is measured in units of Henrys (H) = Wb/A

• The magnetic energy (in Joules) stored by the inductor is expressed as

\[ W_m = \frac{1}{2} LI^2 \]
Inductors and Inductance

- Since we know that magnetic fields produce forces on nearby current elements, and that those magnetic fields can be generated by an isolated or coupled set of current carrying circuits, then it is only reasonable that such circuits may induce fields and magnetization between them.

- We can calculate the individual flux linkage between the two components as

\[
\Phi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}
\]

- Likewise we can determine a mutual inductance between the circuits that is equal from circuit 12 as it is from circuit 21 as

\[
M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \Phi_{12}}{I_2}
\]

- Individual inductances are

\[
L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Phi_1}{I_1}, \quad L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Phi_2}{I_2}
\]

- The total magnetic energy in the circuit is

\[
W_m = W_1 + W_2 + W_{12} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2
\]

8/17/2012
Inductors and Inductance (2)

- As we eluded to before, you should think of an inductor as a conductor shaped in such a way as to store magnetic energy.
- Typical examples include toroids, solenoids, coaxial transmission lines, and parallel-wire transmission lines.
- One can determine the inductance for a given geometry using the following technique
  - Choose a suitable coordinate system
  - Let the inductor carry current, I
  - Determine B from Biot-Savart’s or Amperes Law and calculate the magnetic flux
  - Find L as a function of the flux times the number of turns over the current carried
- Mutual inductance may be calculated by a similar approach
  - Determine the internal inductance, $L_{\text{in}}$ for the flux generated by the first inductor
  - Determine the external inductance, $L_{\text{ext}}$ produced by the flux external of the first inductor
  - The sum of the internal and external inductance equals the individual inductances plus the mutual inductance between the elements

\[
M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \Psi_{12}}{I_2} \quad \Psi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}
\]

- For circuit theory, we can also write the inductance as which provides a very useful equation when quickly mapping out electronic circuits

\[
L_{\text{ext}} C = \mu \varepsilon \quad RC = \frac{\varepsilon}{\sigma}
\]

\[
\Rightarrow R \frac{\mu \varepsilon}{L_{\text{ext}}} = \frac{\varepsilon}{\sigma} \quad \Rightarrow \frac{R}{L_{\text{ext}}} = \frac{1}{\mu \sigma}
\]
Inductors and Inductance (3)

1. Wire
   \[ L = \frac{\mu_0 \ell}{8\pi} \]

2. Hollow cylinder
   \[ L = \frac{\mu_0 \ell}{2\pi} \left( \ln \frac{2\ell}{a} - 1 \right) \]
   \[ \ell \gg a \]

3. Parallel wires
   \[ L = \frac{\mu_0 \ell}{\pi} \ln \frac{d}{a} \]
   \[ \ell \gg d, d \gg a \]

4. Coaxial conductor
   \[ L = \frac{\mu_0 \ell}{\pi} \ln \frac{b}{a} \]

5. Circular loop
   \[ L = \frac{\mu_0 \ell}{2\pi} \left( \ln \frac{4\ell}{d} - 2.45 \right) \]
   \[ \ell = 2\pi \rho_o, \rho_o \gg d \]

6. Solenoid
   \[ L = \frac{\mu_0 N^2 S}{\ell} \]
   \[ \ell \gg a \]

7. Torus (of circular cross section)
   \[ L = \mu_0 N^2 \left[ \rho_o - \sqrt{\rho_o^2 - a^2} \right] \]

8. Sheet
   \[ L = \mu_0 2\ell \left( \ln \frac{2\ell}{b + t} + 0.5 \right) \]
Inductance Example (1)

- Calculate the self inductance per unit length of an infinitely long solenoid

\[
\text{solenoid has N turns} \\
\text{n} = \frac{N}{l} = \text{turns per unit length} \\
B = \mu H = \mu I n \\
\Psi = BS = \mu I n S \\
\lambda' = \frac{\lambda}{l} = n \Psi = \mu I n^2 S \\
L' = \frac{L}{l} = \frac{\lambda'}{I} = \mu n^2 S = \mu n^2 S
\]
Magnetic Energy

- Recall, \( W_E = \frac{1}{2} \int (\vec{E} \cdot \vec{D}) dv = \frac{1}{2} \int \varepsilon E^2 dv \)

- We can derive a similar term for magnetic energy using the relation for energy as a function of inductance

\[
W_m = \frac{1}{2} LI^2
\]

\[
\Delta L = \frac{\Delta \Psi}{\Delta I} = \frac{\mu H \Delta x \Delta y}{\Delta I}
\]

where, \( \Delta I = H \Delta z \)

\[
\Delta W_m = \frac{1}{2} \Delta L \Delta I^2 = \frac{1}{2} \mu H^2 \Delta x \Delta y \Delta z
\]

Assume a wire carrying current along the z direction that generates a field in the x-y plane

\[
\Delta W_m = \frac{1}{2} \mu H^2 \Delta v
\]

\[
w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2 = \frac{1}{2} \left( \vec{H} \cdot \vec{B} \right) = \frac{B^2}{2\mu}
\]

\[
W_m = \int w_m dv = \int \frac{1}{2} \left( \vec{H} \cdot \vec{B} \right) dv = \int \frac{1}{2} \mu H^2 dv
\]
Inductance Example (2 part 1)

- Determine the self inductance of a coaxial cable of inner radius a and outer radius b

$$0 \leq \rho \leq a$$

$$\vec{B}_1 = \frac{\mu l \rho}{2\pi a^2} \hat{\phi}$$

$$d\Psi_1 = B_1 d\rho dz = \frac{\mu l \rho}{2\pi a^2} d\rho dz$$

$$d\lambda_1 = d\Psi_1 \frac{I_{enc}}{I} = \frac{\mu l \rho^2}{2\pi a^2} d\rho dz$$

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{l} \frac{\mu l \rho^3}{2\pi a^4} d\rho dz = \frac{\mu l l}{8\pi}$$

$$L_m = \frac{\lambda_1}{l} = \frac{\mu l}{8\pi} \quad \text{H/m}$$

$$L'_m = \frac{L_m}{l} = \frac{\mu}{8\pi} \quad \text{H/m}$$

$$a \leq \rho \leq b$$

$$\vec{B}_2 = \frac{\mu l \rho}{2\pi b^2} \hat{\phi}$$

$$d\Psi_2 = B_2 d\rho dz = \frac{\mu l}{2\pi b^2} d\rho dz$$

$$d\lambda_2 = d\Psi_2 \frac{I_{enc}}{I} = \frac{\mu l}{2\pi b^2} d\rho dz$$

$$\lambda_2 = \int_{\rho=0}^{a} \int_{z=0}^{l} \frac{\mu l \rho}{2\pi b^4} d\rho dz = \frac{\mu l l}{2\pi} \ln \left( \frac{b}{a} \right)$$

$$L_{ext} = \frac{\lambda_2}{l} = \frac{\mu l}{2\pi} \ln \left( \frac{b}{a} \right) \quad \text{H/m}$$

$$L'_{ext} = \frac{L_m}{l} = \frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right) \quad \text{H/m}$$

Total Inductance at b

$$L = L_m + L_{ext} = \frac{\mu}{2\pi} \left[ \frac{1}{4} + \ln \left( \frac{b}{a} \right) \right] \quad \text{H/m}$$
Inductance Example (2 part 2)

- Alternatively we can solve the same problem form a magnetic energy perspective
- Determine the self inductance of a coaxial cable of inner radius $a$ and outer radius $b$

\[ L_{\text{in}} = \frac{2}{I^2} \int \frac{B_1^2}{2\mu} \, dv = \frac{2}{I^2} \int \frac{I^2 \rho^2}{\mu 8\pi^2 a^4} \, dv = \mu \int \frac{\rho^2}{4\pi^2 a^4} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu l}{8\pi} \]

\[ L_{\text{ext}} = \frac{2}{I^2} \int \frac{B_2^2}{2\mu} \, dv = \frac{2}{I^2} \int \frac{1}{2\mu} \left( \frac{\mu l}{2\pi \rho} \right)^2 \, dv = \mu \int \frac{1}{4\pi^2 \rho^2} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu l}{2\pi} \ln \left( \frac{b}{a} \right) \]

\[ L = L_{\text{in}} + L_{\text{ext}} = \frac{\mu l}{2\pi} \left[ \frac{1}{4} + \ln \left( \frac{b}{a} \right) \right] \text{H/m} \]
Inductance Example (3)

- Determine the inductance of a two wire transmission line with wire radius $a$ and separation distance $d$

**Method 1**

$$0 \leq \rho \leq a$$

$$\vec{B}_1 = \frac{\mu Il}{2\pi a^2} \hat{a}_\phi$$

$$\lambda_1 = \frac{\mu Il}{8\pi}$$

Same as sol. for center wire in a coax

$$a \leq \rho \leq d - a$$

$$\bar{B}_2 = \frac{\mu Il}{2\pi \rho} \hat{a}_\phi$$

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{d-a} \int_{z=0}^{l} \frac{\mu Il}{2\pi \rho} d\rho d\zeta = \frac{\mu Il}{2\pi} \ln\left(\frac{d-a}{a}\right)$$

$$\lambda_1 + \lambda_2 = \frac{\mu Il}{2\pi} \left[ \frac{1}{4} + \ln\left(\frac{d-a}{a}\right) \right]$$

$$\lambda = 2(\lambda_1 + \lambda_2) = \frac{\mu Il}{\pi} \left[ \frac{1}{4} + \ln\left(\frac{d-a}{a}\right) \right] = LI$$

$$d >> a, \text{then}$$

$$L' = \frac{L}{l} = \frac{\mu l}{\pi} \left[ \frac{1}{4} + \ln\left(\frac{d-a}{a}\right) \right]$$
• Determine the inductance of a two wire transmission line with wire radius $a$ and separation distance $d$

\[
0 \leq \rho \leq a
\]

\[
\tilde{B}_1 = \frac{\mu l \rho}{2 \pi a^2} \hat{a}_\phi
\]

\[
a \leq \rho \leq b
\]

\[
\tilde{B}_2 = \frac{\mu l}{2 \pi \rho} \hat{a}_\phi
\]

\[
W_m = \frac{1}{2} LI^2 = \int \frac{1}{2} \mu (\vec{H} \cdot \vec{B}) \, dv = \int \frac{B^2}{2 \mu} \, dv
\]

\[
L_{in} = \frac{2}{I^2} \int \frac{B_1^2}{2 \mu} \, dv = \frac{2}{I^2} \int \frac{I^2 \rho^2}{\mu 8 \pi^2 a^4} \, dv = \mu \int \frac{\rho^2}{4 \pi^2 a^4} d\rho d\phi dz = \frac{\mu l}{8 \pi}
\]

\[
L_{ext} = \frac{2}{I^2} \int \frac{B_2^2}{2 \mu} \, dv = \frac{2}{I^2} \int \frac{1}{2 \mu} \left( \frac{\mu l}{2 \pi \rho} \right)^2 \, dv = \mu \int \frac{1}{4 \pi^2 \rho^2} d\rho d\phi dz = \frac{\mu l}{2 \pi} \ln\left( \frac{d-a}{a} \right)
\]

\[
L = 2(L_{in} + L_{ext}) = \frac{\mu l}{\pi} \left[ \frac{1}{4} + \ln\left( \frac{d-a}{a} \right) \right] H / m
\]
Inductance Example (4)

- Two coaxial circular wires of radius $a$ and $b$ are separated by a distance $h \gg a, b$. Find the mutual inductance between the wires.

Using the equation for a magnetic dipole

$$\vec{A} = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r} = \frac{\mu_0 \vec{m} \times \vec{r}}{4\pi r^3}$$

where

$$\vec{m} = I \pi a^2 \hat{z}$$

$$\hat{a}_z \times \hat{a}_r = (\sin \theta) \hat{a}_\phi$$

$$\vec{A}_1 = \frac{\mu I_1 \pi a^2 \sin \theta}{4\pi r^2} \hat{a}_\phi = \frac{\mu I_1 a^2 b \sin \theta}{4(h^2 + b^2)^{3/2}} \hat{a}_\phi$$

$h \gg b$

$$\vec{A}_1 = \frac{\mu I_1 a^2 b}{4h^3} \hat{a}_\phi$$

$$\Psi_{12} = \oint \vec{A}_1 \cdot d\vec{l}_2 = \frac{\mu I_1 a^2 b}{4h^3} 2\pi b = \frac{\mu I_1 \pi a^2 b^2}{2h^3}$$

$$M_{12} = \frac{\Psi_{12}}{I_1} = \frac{\mu \pi a^2 b^2}{2h^3}$$
The following relations allow one to solve magnetic field problems in a manner similar to that of electronic circuits.

- It provides a clear means of designing transformers, motors, generators, and relays using a lumped circuit model.
- The analogy between electronic and magnetic circuits is provided below.

### Table 8.4 Analogy between Electric and Magnetic Circuits

<table>
<thead>
<tr>
<th>Electric</th>
<th>Magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conductivity $\sigma$</td>
<td>Permeability $\mu$</td>
</tr>
<tr>
<td>Field intensity $E$</td>
<td>Field intensity $H$</td>
</tr>
<tr>
<td>Current $I = \int \mathbf{J} \cdot d\mathbf{S}$</td>
<td>Magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$</td>
</tr>
<tr>
<td>Current density $J = \frac{I}{S} = \sigma E$</td>
<td>Flux density $B = \frac{\Psi}{S} = \mu H$</td>
</tr>
<tr>
<td>Electromotive force (emf) $V$</td>
<td>Magnetomotive force (mmf) $\mathcal{F}$</td>
</tr>
<tr>
<td>Resistance $R$</td>
<td>Reluctance $\mathcal{R}$</td>
</tr>
<tr>
<td>Conductance $G = \frac{1}{R}$</td>
<td>Permeance $\mathcal{P} = \frac{1}{\mathcal{R}}$</td>
</tr>
<tr>
<td>Ohm’s law $R = \frac{V}{I} = \frac{\ell}{\sigma S}$</td>
<td>Ohm’s law $\mathcal{R} = \frac{\mathcal{F}}{\Psi} = \frac{\ell}{\mu S}$</td>
</tr>
<tr>
<td>or $V = E\ell = IR$</td>
<td>or $\mathcal{F} = H\ell = \Psi\mathcal{R} = NI$</td>
</tr>
<tr>
<td>Kirchhoff’s laws: $\Sigma I = 0$</td>
<td>Kirchhoff’s laws: $\Sigma \Psi = 0$</td>
</tr>
<tr>
<td>$\Sigma V - \Sigma RI = 0$</td>
<td>$\Sigma \mathcal{F} - \Sigma \mathcal{R} \Psi = 0$</td>
</tr>
</tbody>
</table>
To develop our model we define a magnetomotive (mmf) force that is equivalent to voltage for electronic circuits

\[ \mathcal{F} = NI = \oint \vec{H} \cdot d\vec{l} \]

We then define a reluctance which is equivalent to magnetic resistance

\[ \mathcal{R} = \frac{l}{\mu S} \]

And show that the magnetic flux is equivalent to current using the following Ohmic style relation

\[ \mathcal{F} = \Psi \mathcal{R} \]
Because these are often mechanical system, it is extremely useful to be able to determine the forces generated by these circuits and those required to move components from one location to another.

For ease of use, we will ignore fringe fields in our calculations.

Also, by using ferromagnetic materials and applying simple magnetic boundary conditions, we path very strong fields into specific geometric shapes. This allows one to focus the force to a specific location and move an object in a well defined manner.

Let's examine the force required to pull a magnetic bar vertically up to an electromagnetic yoke.

\[
dW_m = -F_T dl = 2 \left( \frac{1}{2} \frac{B^2}{\mu_0} S dl \right)
\]

\[
F_T = -2 \left( \frac{B^2 S}{2 \mu_0} \right) = F_{1/2} + F_{1/2}
\]

\[
F_{1/2} = -\frac{B^2 S}{2 \mu_0}
\]

Force on a single gap

\[
p_{1/2} = \frac{F_{1/2}}{A} = \frac{B^2}{2 \mu_0} = \frac{1}{2} BH = w_m
\]

Note: This diagram shows a yoke pulling a bar magnet (keeper) at two gap locations.

pressure = energy density!!!!!!
A steel toroidal core with permeability, $\mu = \mu_r \mu_0$, has a mean radius, $\rho_0$, and a circular cross section of diameter $2a$. Calculate the current required to generate a flux, $\Psi$, in the core.

\[ B = \frac{\mu NI}{l} = \frac{\mu_0 \mu_r NI}{2\pi \rho_0} \]

\[ \Psi = \int B dS = \frac{\mu_0 \mu_r NI}{2\pi \rho_0} \pi a^2 \]

\[ I = \frac{2\rho_0 \Psi}{N \mu_0 \mu_r a^2} \]

\[ Z = NI = \Psi R = \Psi \frac{l}{\mu S} = \Psi \frac{2\pi \rho_0}{\mu_0 \mu_r \pi a^2} = \frac{\mu_0 \mu_r NI 2\pi \rho_0}{\mu_0 \mu_r \pi a^2 2\pi \rho_0} \pi a^2 = NI \]

\[ I = \frac{\Psi R}{N} = \frac{\Psi}{N} \frac{2\pi \rho_0}{\mu_0 \mu_r \pi a^2} = \frac{2\rho_0 \Psi}{N \mu_0 \mu_r a^2} \]
A steel toroidal core with permeability $\mu = \mu_r \mu_0$ has a mean radius $\rho_0$, and a rectangular cross section, $2a \times b$. Calculate the current required to generate a flux, $\Psi$, in the core.

![Toroid Diagram]

**Force on Magnetic Materials (Toroid with Rectangular Cross-section)**

- $B = \frac{\mu NI}{l} = \frac{\mu_0 \mu_r NI}{2\pi \rho_0}$

- $\Psi = \int B dS = \int_{\rho_0^{-a}}^{\rho_0^{+a}} \frac{\mu_0 \mu_r NI}{2\pi \rho} \, b \, d\rho = \frac{\mu_0 \mu_r N b}{2\pi} \int_{\rho_0^{-a}}^{\rho_0^{+a}} \frac{d\rho}{\rho} = \frac{\mu_0 \mu_r N b}{2\pi} \ln \left( \frac{\rho_0 + a}{\rho_0 - a} \right)$

- $L = \frac{N\Psi}{I} = \frac{\mu_0 \mu_r N^2 b}{2\pi} \ln \left( \frac{\rho_0 + a}{\rho_0 - a} \right)$

- $W_m = \frac{1}{2} LI^2 = \frac{\mu_0 \mu_r N^2 b I^2}{4\pi} \ln \left( \frac{\rho_0 + a}{\rho_0 - a} \right)$
For the magnetic circuit shown below, with magnetic flux density of 1.5 Wb/m² and a relative permeability of 50.

- Find the individual reluctances and determine the total current required to generate a particular flux value. All branches have a cross sectional area of 0.001 cm²

\[ \mu_r = 50 \]

\[ \mathcal{R}_1 \Rightarrow \text{path}143 \]

\[ \mathcal{R}_2 \Rightarrow \text{path}123 \]

\[ \mathcal{R}_3 \Rightarrow \text{path}16\text{and}35 \]

\[ \mathcal{R}_a \Rightarrow \text{path}56 \]

\[ \mathcal{R}_1 = \mathcal{R}_2 = \frac{l}{\mu_0 \mu_r S} = \frac{(30 \times 10^{-4})}{\mu_0 (50)(10 \times 10^{-4})} = \frac{3 \times 10^8}{20\pi} \]

\[ \mathcal{R}_3 = \frac{(90 \times 10^{-4})}{\mu_0 (50)(10 \times 10^{-4})} = \frac{0.9 \times 10^8}{20\pi} \]

\[ \mathcal{R}_a = \frac{l}{\mu_0 S} - \frac{(10 \times 10^{-4})}{\mu_0 (10 \times 10^{-4})} = \frac{5 \times 10^8}{20\pi} \]

\[ \mathcal{R}_1 \parallel \mathcal{R}_2 = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} \]

\[ \mathcal{R}_T = \mathcal{R}_3 + \mathcal{R}_a + \mathcal{R}_1 \parallel \mathcal{R}_2 = \frac{7.4 \times 10^8}{20\pi} \]

\[ \mathcal{Z} = NI = \Psi_a \mathcal{R}_T \]

\[ \Psi = B_a S \]

\[ I = \frac{B_a S \Psi_T}{N} = \frac{1.5 \times (10 \times 10^8)(7.4 \times 10^8)}{400 \times 20\pi} = 44.16A \]
A U-shaped electromagnetic is designed to lift a 400 kg mass (which includes the mass of the keeper). The iron yoke with relative permeability of 3000 has a cross section of 40 cm$^2$ and a mean length of 50 cm. Each of the air gaps are 0.1mm long. Neglecting the reluctance of the keeper, calculate the number of turns in the coil when the excitation current is 1A.

\[
W_m = \frac{1}{2} \int \mu |H|^2 \, dv = \frac{1}{2 \mu_0} \int |\vec{B}|^2 \, dv = \int \vec{F} \cdot d\vec{y}
\]

\[
dW_m = dW_{m, \text{air}_\text{gap}} = 2 \frac{|\vec{B}|^2}{2 \mu_0} S \, dy = F \, dy
\]

\[
F = 2 \frac{B_a^2 S}{2 \mu_0} = mg
\]

\[
B_a = \sqrt{\frac{mg \mu_0}{S}} = 1.1 \text{ Wb/m}^2
\]

\[
\mathcal{F} = NI = \Psi \left( \mathcal{R}_y + \mathcal{R}_g \right)
\]

\[
\mathcal{R}_y = \frac{l}{\mu_0 \mu_y S} = \frac{2 \times (0.50)}{\mu_0 3000 (0.004)} = \frac{6 \times 10^6}{48 \pi}
\]

\[
\mathcal{R}_g = \frac{l}{\mu_0 S} = \frac{2 \times (0.0001)}{\mu_0 (0.004)} = \frac{5 \times 10^6}{48 \pi}
\]

\[
\mathcal{F}_g = \frac{\mathcal{R}_g}{\mathcal{R}_y + \mathcal{R}_g} \mathcal{F} = \frac{\mathcal{R}_g}{\mathcal{R}_y + \mathcal{R}_g} NI = \frac{6}{5 + 6} NI
\]

\[
\mathcal{F}_g = \frac{\mathcal{R}_g}{\mathcal{R}_y + \mathcal{R}_g} NI = H_a l_a = \frac{B_a l_a}{\mu_0}
\]

\[
N = \frac{1.11 (0.0001) 11}{\mu_0 5 (1)} = 162
\]
Force on Magnetic Materials (Example 3b)

- Yoke pulling a keeper from both ends

\[ W_m = \frac{1}{2} \int_y \left( \vec{H} \cdot \vec{B} \right) \, dv = \frac{1}{2} \mu_0 \int_y B^2 \, dv = \int_L \vec{F} \cdot d\vec{y} \]

\[ dW_m = dW_{m, \text{air\_gap}} = 2 \frac{B^2}{2 \mu_0} S \, dy = \frac{\zeta}{2 \mu_0 S} \, dy = F \, dy \]

\[ F_i = \frac{\partial W_m}{\partial x} = 2 \frac{B_a^2 S}{2 \mu_0} = mg \]

\[ B_a = \sqrt{\frac{mg \mu_0}{S}} \]

\[ W_m = \frac{1}{2} L i_a^2 (t) \]

\[ L = \frac{N \zeta}{i_a (t)} = \frac{\Psi}{i_a (t)} \]

\[ \zeta = NI = \Psi \sum \mathcal{R}_i \]

\[ \mathcal{R}_y = \frac{l_y}{\mu_0 \mu_r S}, \quad \mathcal{R}_k = \frac{l_k}{\mu_0 \mu_r k S}, \quad \mathcal{R}_g = 2 \frac{y(t)}{\mu_0 S} \]

\[ \zeta_g = \frac{\mathcal{R}_g}{\mathcal{R}_y + \mathcal{R}_g + \mathcal{R}_k} \quad \zeta = \frac{\mathcal{R}_g}{\mathcal{R}_y + \mathcal{R}_g + \mathcal{R}_k} NI \]

\[ L(y) = \frac{N^2}{\sum \mathcal{R}_i} \]

\[ F_i = \frac{\partial W_m}{\partial x} = \frac{1}{2} \frac{\partial L(y)}{\partial y} = \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{N^2}{\sum \mathcal{R}_i} \right) \]

8/17/2012
Force on Magnetic Materials
(Example 3c)

- Yoke pulling a keeper by gravitational force

\[ F_i = \frac{\partial W_m}{\partial x} = 2 \frac{B_a^2 S}{2 \mu_0} = mg \]

\[ B_a = \sqrt{\frac{mg \mu_0}{S}} \]

\[ W_m = \frac{1}{2} L i_a^2(t) \]

\[ L = \frac{N \Phi}{i_a(t)} = \frac{\Psi}{i_a(t)} \]

\[ \Im = NI = \Psi \sum R_i \]

\[ R_y = \frac{l_y}{\mu_0 \mu_r S}, R_k = \frac{l_k}{\mu_0 \mu_r k S}, R_g = 2 \frac{x(t)}{\mu_0 S} \]

\[ \Im_g = \frac{R_g}{R_y + R_g + R_k} \]

\[ L(x(t)) = \sum R_i = \frac{N^2 \mu_0 \mu_r \mu_{rk} S}{\mu_{rk} l_y + 2 \mu_r \mu_{rk} x(t) + \mu_{rk} l_k} \]

\[ F_i = \frac{\partial W_m}{\partial x} = \frac{1}{2} \frac{\partial (L(x(t)) i_a^2(t))}{\partial x} = -\frac{N^2 \mu_0 \mu_{ry}^2 \mu_{rk}^2 S^2 i_a^2(t)}{\left(\mu_{rk} l_y + 2 \mu_{ry} \mu_{rk} x(t) + \mu_{ry} l_k\right)^2} = 2 \frac{B_a^2 S}{2 \mu_0} = mg \]

\[ \frac{dx}{dt} = v \]

\[ \frac{dv}{dt} = \frac{1}{m} \left[ -\frac{N^2 \mu_0 \mu_{ry}^2 \mu_{rk}^2 S^2 i_a^2(t)}{\left(\mu_{rk} l_y + 2 \mu_{ry} \mu_{rk} x(t) + \mu_{ry} l_k\right)^2} - mg \right] \]
Force on Magnetic Materials
(Example 3d)

- Yoke pulling a spring loaded keeper from both ends

\[ F_l = \frac{\partial W_m}{\partial x} = 2 \frac{B_a^2 S}{2 \mu_0} = k_s x - k_{s1} x - k_{s2} x \]

\[ B_a = \sqrt{\frac{mg \mu_0}{S}} \]

\[ W_m = \frac{1}{2} L i_a^2(t) \]

\[ L = \frac{N \Xi}{i_a(t)} = \Psi \]

\[ \Xi = NI = \Psi \sum R_i \]

\[ R_y = \frac{l_y}{\mu_0 \mu_r S}, \quad R_k = \frac{l_k}{\mu_0 \mu_r k S}, \quad R_g = 2 \frac{x(t)}{\mu_0 S} \]

\[ \Xi_g = \frac{R_g}{R_y + R_g + R_k} \Xi = \frac{R_g}{R_y + R_g + R_k} NI \]

\[ L(x(t)) = \frac{N^2}{\sum R_i} = \frac{N^2 \mu_0 \mu_r \mu_k S}{\mu_k l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_k} \]

\[ F_l = \frac{\partial W_m}{\partial x} = \frac{1}{2} \frac{\partial (L(x(t)) \ast i_a^2(t))}{\partial x} = -\frac{N^2 \mu_0 \mu_r^2 \mu_k^2 S^2 i_a(t)}{\left(\mu_k l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_k\right)^2} = \frac{2 B_a^2 S}{2 \mu_0} = k_s x - k_{s1} x - k_{s2} x \]

\[ \frac{dx}{dt} = v \]

\[ \frac{dv}{dt} = \frac{1}{m} \left( -\frac{N^2 \mu_0 \mu_r^2 \mu_k^2 S^2 i_a(t)}{\left(\mu_k l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_k\right)^2} - k_s x + k_{s1} x + k_{s2} x \right) \]
The system is driven by voltage control and not steady state current

\[ F_I = \frac{\partial W_m}{\partial x} = \frac{1}{2} \frac{\partial (L(x(t)) \cdot i_a^2(t))}{\partial x} = - \frac{N^2 \mu_0 \mu_{ry} \mu_r k^2 S^2 i_a^2(t)}{\left( \mu_r l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_y \right)^2} = \frac{B_s^2 S}{2 \mu_0} = k_s x - k_{s1} x - k_{s2} x \]

\[ u_a(t) = R i_a(t) + \frac{d \Psi}{dt} = R i_a(t) + \frac{d (L(x) \cdot i_a(t))}{dt} \]

\[ u_a(t) = R i_a(t) + \frac{d i_a(t)}{dt} + i_a \frac{d L(x) \cdot dx}{dt} \]

\[ \Rightarrow \frac{d i_a(t)}{dt} = \frac{1}{L(x)} \left( - R_i_a(t) + \frac{2N^2 \mu_0 \mu_{ry} \mu_r k^2 S^2 i_a^2(t)}{\left( \mu_r l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_y \right)^2} i_a v(t) + u_a(t) \right) \]

Thus the complete set of equations is:

\[ \frac{d i_a(t)}{dt} = \frac{1}{L(x)} \left( - R_i_a(t) + \frac{2N^2 \mu_0 \mu_{ry} \mu_r k^2 S^2 i_a^2(t)}{\left( \mu_r l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_y \right)^2} i_a v(t) + u_a(t) \right) \]

\[ \frac{dv}{dt} = \frac{1}{m} \left( - \frac{N^2 \mu_0 \mu_{ry} \mu_r k^2 S^2 i_a^2(t)}{\left( \mu_r l_y + 2 \mu_r \mu_r k x(t) + \mu_r l_y \right)^2} - k_s x + k_{s1} x + k_{s2} x \right) \]

\[ \frac{dx}{dt} = v \]