

4.0 AMBIGUITY FUNCTION

4.1 INTRODUCTION

We now turn our attention to the ambiguity function. The ambiguity function, which is denoted as $|\chi(\tau, f)|$, is primarily used to gain an understanding of how a signal processor responds, or reacts, to a given returned signal. As indicated in the notation, the independent variables of the ambiguity function are time (τ) and frequency (f). The time variable is normally associated with target range and the frequency variable is normally associated with target Doppler. The fact that the notation contains a magnitude sign (i.e. $|\ |$) is used to indicate that we are characterizing the amplitude of the signal processor output.

In a strict sense, when one uses the phrase ambiguity function there is an underlying assumption that the signal processor is matched to the transmitted waveform. If the signal processor is not matched to the transmitted waveform, the proper terminology in the ambiguity function context would be to refer to the cross ambiguity function. In practice we don't always make the distinction and simply use the phrase ambiguity function.

We want to derive the general equation for the (cross) ambiguity function and then show how to derive specific ambiguity function expressions for some simple waveforms and signal processors. Following this we will discuss two representations that are suitable for numerically computing the ambiguity function using the Fast Fourier Transform (FFT). This will allow generation of ambiguity functions of advanced waveforms and signal processors.

4.1 AMBIGUITY FUNCTION DEVELOPMENT

Since the ambiguity function can be thought of as the response of a signal processor to a received radar waveform, this is the approach we take in deriving the ambiguity function.

Let the transmitted waveform be represented by the base-band signal, $u(t)$.¹ The normalized (base-band) signal received from a (constant range-rate) target at a range R and range-rate \dot{R} is given by

$$v_R(t) = u(t - \tau_R) e^{j2\pi f_d t} \quad (4-1)$$

¹ Recall that if an actual signal is $v_{RF}(t) = A(t) \cos(\omega_{RF} t - \phi(t))$ then the base-band representation of that signal is $v_B(t) = A(t) e^{j\phi(t)}$. That is, the base-band signal captures the amplitude and phase modulation of the actual signal in the form of a complex variable. Base-band notation is also termed complex signal notation.

where $\tau_R = 2R/c$ is the range delay and $f_d = -2\dot{R}/\lambda$ is the target Doppler frequency. λ is the wavelength of the transmitted signal and c is the speed of light.

The signal processor configuration we use in deriving the ambiguity function is shown in *Figure 4-1*. In this figure, $h(t)$ is a low-pass function and f_s is thought of as the frequency to which the signal processor is tuned. Thus, the overall signal processor is a band-pass filter centered at f_s . This is indicated in *Figure 4-2* where the left plot is the frequency response of $h(t)$ and the right plot is the frequency response of the signal processor.

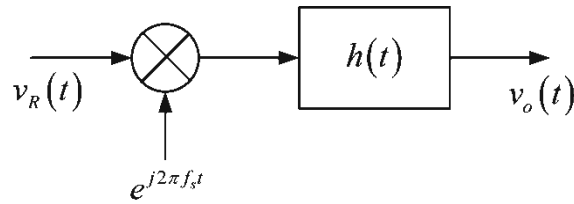


Figure 4-1 – Signal Processor

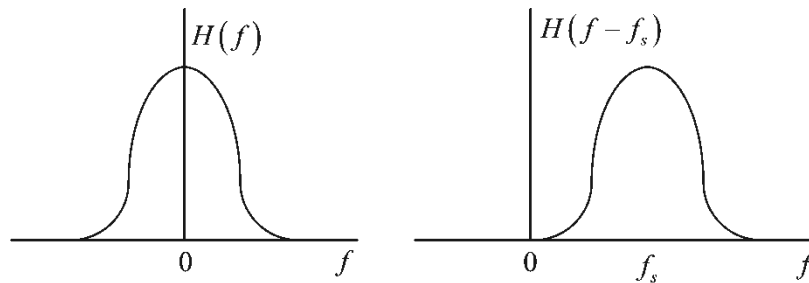


Figure 4-2 – Signal Processor Frequency Response

In keeping with the concept of matched filters, we normally define $h(t)$ in terms of the waveform to which it is matched. Specifically, we use

$$h(t) = Kv^*(t_0 - t) \quad (4-2)$$

and set $K = 1$ and $t_0 = 0$ to get

$$h(t) = v^*(-t). \quad (4-3)$$

In Equation (4-3), $v(t)$ is the waveform to which the signal processor is matched.

With the above, the impulse response of the signal processor is

$$h_{SP}(t) = e^{j2\pi f_s t} h(t) = e^{j2\pi f_s t} v^*(-t). \quad (4-4)$$

The output of the signal processor is the convolution of $v_R(t)$ and $h_{SP}(t)$ or

$$v_o(\gamma) = \int_{-\infty}^{\infty} v_R(\alpha) h_{SP}(\gamma - \alpha) d\alpha. \quad (4-5)$$

Using Equations (4-1) and (4-4) yields

$$\begin{aligned} v_o(\gamma) &= \int_{-\infty}^{\infty} u(\alpha - \tau_R) e^{j2\pi f_d \alpha} e^{j2\pi f_s(\gamma - \alpha)} v^*(\alpha - \gamma) d\alpha \\ &= e^{j2\pi f_s \gamma} \int_{-\infty}^{\infty} u(\alpha - \tau_R) v^*(\alpha - \gamma) e^{j2\pi(f_d - f_s)\alpha} d\alpha \end{aligned} \quad (4-6)$$

If we make the change of variable $t = \alpha - \tau_R$ Equation (4-6) can be written as

$$\begin{aligned} v_o(\gamma) &= \int_{-\infty}^{\infty} u(t) v^*(t + \tau_R - \gamma) e^{j2\pi(f_d - f_s)(t + \tau_R)} dt \\ &= e^{j2\pi f_s \tau} e^{j2\pi f_d \tau_R} \int_{-\infty}^{\infty} u(t) v^*(t + \tau) e^{j2\pi f t} dt \end{aligned} \quad (4-7)$$

where we have made the substitutions $\tau = \tau_R - \gamma$ and $f = f_d - f_s$.

The variables τ and f are often termed the mismatched range and Doppler of the ambiguity function. More specifically, τ is the difference between the target range delay and the time at which we look at the signal processor output. If $\tau = 0$, we say that we are at matched range. That is, we are looking at the signal processor at a time equal to the time delay of the target. f is the difference between the target Doppler frequency and the frequency to which the signal processor is matched. If $f = 0$ we say that the signal processor is matched to the target Doppler, or vice-versa. In this case we say that we are at matched Doppler.

Since τ and f are the variables of interest, we write Equation (4-7) in terms of them and change the dependent variable from $v_o(\gamma)$ to $\chi(\tau, f)$. Thus, we get

$$\chi(\tau, f) = e^{j2\pi f_s \tau} e^{j2\pi f_d \tau_R} \int_{-\infty}^{\infty} u(t) v^*(t + \tau) e^{j2\pi f t} dt. \quad (4-8)$$

Finally, if we take the absolute value of Equation (4-8) we get the ambiguity function or

$$|\chi(\tau, f)| = \left| \int_{-\infty}^{\infty} u(t) v^*(t + \tau) e^{j2\pi f t} dt \right|. \quad (4-9)$$

We often attribute special names to plots of $|\chi(\tau, f)|$ for specific values of τ and f . In particular:

- If we let $f = 0$ to yield $|\chi(\tau, 0)|$ we say that we have the matched-Doppler, range cut of the ambiguity function. This is what we normally think of as the output of the classical matched filter.

- If we let $\tau = 0$ to yield $|\chi(0, f)|$ we say that we have the matched-range, Doppler cut of the ambiguity function.
- If we let $f = f_k$ to yield $|\chi(\tau, f_k)|$ we say that we have a range cut at some mismatched Doppler of f_k .
- If we let $\tau = \tau_k$ to yield $|\chi(\tau_k, f)|$ we say that we have a Doppler cut at some mismatched range of τ_k .

As we discuss in Chapter 5, the ambiguity function provides a wealth of information about radar waveforms and how they interact with the environment and the radar signal processor.

4.3 EXAMPLE – UNMODULATED PULSE

To begin the study of specific ambiguity functions we want to derive the equation for the ambiguity function of an unmodulated pulse of width τ_p . We will assume that the signal processor is matched to the transmitted pulse. With this we can write

$$v(t) = u(t) = \text{rect}\left[\frac{t}{\tau_p}\right] \quad (4-10)$$

$$\text{where } \text{rect}[x] = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

If we substitute this into Equation (4-9) we get

$$|\chi(\tau, f)| = \left| \int_{-\infty}^{\infty} \text{rect}\left[\frac{t}{\tau_p}\right] \text{rect}\left[\frac{t+\tau}{\tau_p}\right] e^{j2\pi ft} dt \right|. \quad (4-11)$$

As we did for the matched filter derivation, we need to consider several regions of τ . To see this we refer to *Figures 4-3* and *4-4* below.

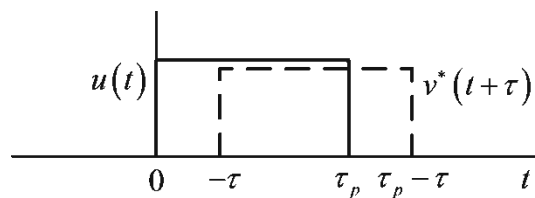


Figure 4-3 – Plot of $u(t)$ and $v^*(t+\tau)$ for $\tau < 0$

In these figures $u(t) = \text{rect}\left[\frac{t}{\tau_p}\right]$ and $v^*(t+\tau) = \text{rect}\left[\frac{t+\tau}{\tau_p}\right]$. It should be clear that for $|\tau| > \tau_p$ the rect functions will not overlap and the integral of Equation (4-11) will be zero. This leads to the observation that $|\chi(\tau, f)| = 0$ for $|\tau| > \tau_p$. We will account for this by multiplying $|\chi(\tau, f)|$ by $\text{rect}\left[\tau/2\tau_p\right]$, where, in this case

$$\text{rect}[x] = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \quad (4-12)$$

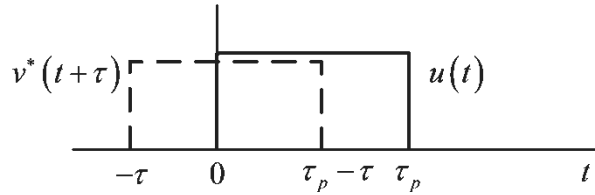


Figure 4-3 – Plot of $u(t)$ and $v^*(t+\tau)$ for $\tau > 0$

For $\tau < 0$ the rect functions overlap from $-\tau$ to τ_p . Thus the ambiguity function becomes

$$|\chi(\tau, f)| = \left| \int_{-\tau}^{\tau_p} e^{j2\pi ft} dt \right| = \left| \frac{e^{j2\pi f\tau_p} - e^{-j2\pi f\tau}}{j2\pi f} \right| \quad (4-13)$$

If we factor $e^{j\pi f(\tau_p - \tau)}$ out of both terms on the far right side the result will be

$$|\chi(\tau, f)| = \left| e^{j\pi f(\tau_p - \tau)} \left(\frac{e^{j\pi f(\tau_p + \tau)} - e^{-j\pi f(\tau_p + \tau)}}{j2\pi f} \right) \right| = |\tau_p + \tau| \left| \text{sinc}(f(\tau_p + \tau)) \right| \quad (4-14)$$

Finally, if we recognize that, since $\tau < 0$, $|\tau| = -\tau$ Equation (4-14) can be written as

$$|\chi(\tau, f)| = |\tau_p - |\tau|| \left| \text{sinc}(f(\tau_p - |\tau|)) \right| \quad (4-15)$$

For $\tau \geq 0$ the rect functions overlap from 0 to $\tau_p - \tau$. Thus the ambiguity function becomes

$$|\chi(\tau, f)| = \left| \int_0^{\tau_p - \tau} e^{j2\pi ft} dt \right| = \left| \frac{e^{j2\pi f(\tau_p - \tau)} - 1}{j2\pi f} \right| \quad (4-16)$$

If we again factor $e^{j\pi f(\tau_p - \tau)}$ out of both terms on the far right side, the result will be

$$|\chi(\tau, f)| = \left| e^{j\pi f(\tau_p - \tau)} \left(\frac{e^{j\pi f(\tau_p - \tau)} - e^{-j\pi f(\tau_p - \tau)}}{j2\pi f} \right) \right| = |\tau_p - \tau| \left| \text{sinc}(f(\tau_p - \tau)) \right|. \quad (4-17)$$

Finally, if we recognize that, since $\tau \geq 0$, $|\tau| = \tau$ Equation (4-17) can be written as

$$|\chi(\tau, f)| = |T - |\tau|| \left| \text{sinc}(f(T - |\tau|)) \right| \quad (4-18)$$

which is the same result we obtained for $\tau < 0$.

If we combine all of the above, we arrive at our final answer of

$$|\chi(\tau, f)| = (\tau_p - |\tau|) \left| \text{sinc}(f(\tau_p - |\tau|)) \right| \text{rect} \left[\frac{\tau}{2\tau_p} \right]. \quad (4-19)$$

We note that the matched-Doppler, range cut, which we obtain by setting $f = 0$, is the same form as the matched filter output that we found in Chapter 3. Specifically,

$$|\chi(\tau, 0)| = (\tau_p - |\tau|) \text{rect} \left[\frac{\tau}{2\tau_p} \right]. \quad (4-20)$$

A sketch of $|\chi(\tau, 0)|$ is shown in *Figure 4-5*.

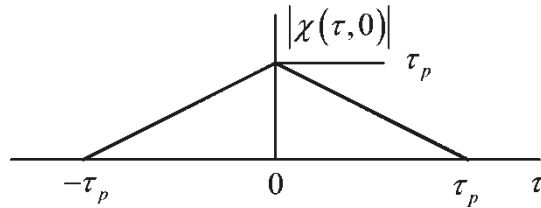


Figure 4-5 – Matched-Doppler Range Cut for an Unmodulated Pulse

The matched-Range, Doppler cut, which we obtain by setting $\tau = 0$, is

$$|\chi(0, f)| = \tau_p \left| \text{sinc}(f\tau_p) \right|. \quad (4-21)$$

A plot of this function is shown in *Figure 4-6*.

Finally, a plot of the center portion of the ambiguity function is shown in *Figure 4-7*.

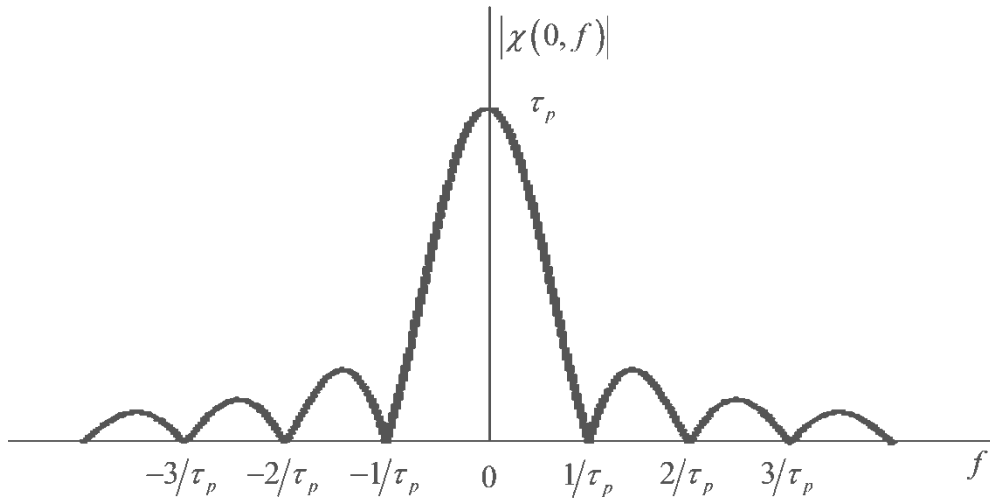


Figure 4-6 – Matched-Range Doppler Cut of an Unmodulated Pulse

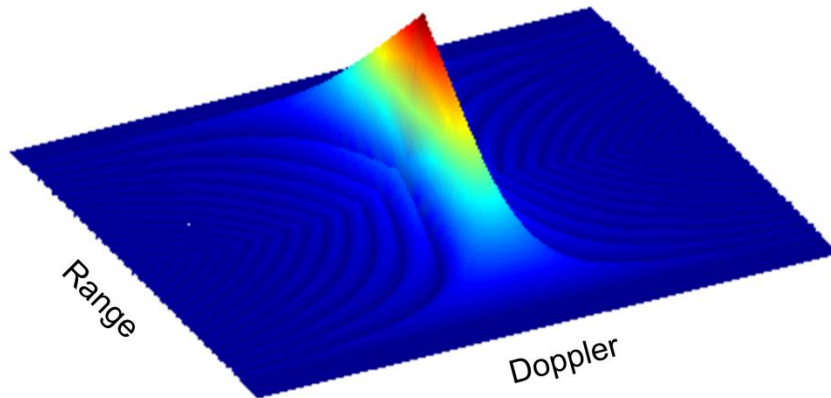


Figure 4-7 – Ambiguity Function Plot of an Unmodulated Pulse

4.4 NUMERICAL TECHNIQUES

While the above approach is suitable for deriving the ambiguity function of simple waveforms and signal processors, it becomes very tedious for complex waveforms and signal processors. Because of this we investigate numerical techniques for generating the ambiguity function. These techniques rely on the fact that modern computers are very fast and that efficient software such as MATLAB and Mathcad is available.

For the first technique we develop the ambiguity function as a sequence of range cuts. If we recall the general representation of the ambiguity function we note that $\chi(\tau, f)$ can be interpreted as the correlation of $u(\tau)e^{j2\pi f\tau}$ with $v(\tau)$. That is,

$$\chi(\tau, f) = \int_{-\infty}^{\infty} u(t)v^*(t+\tau)e^{j2\pi ft} dt = (u(\tau)e^{j2\pi f\tau}) \otimes v(\tau). \quad (4-22)$$

From Fourier transforms, we recognize that

$$\mathfrak{T}\{\chi(\tau, f)\} = \mathfrak{T}\{u(\tau)e^{j2\pi f\tau}\}\mathfrak{T}^*\{v(\tau)\}, \quad (4-23)$$

or

$$F_\chi(\theta) = F_u(\theta)F_v^*(\theta) \quad (4-24)$$

where the definitions of the terms is obvious.

In the algorithm discussed in the next section, we use the FFT to approximate the two Fourier transforms on the right of Equation (4-24). We then perform the indicated multiplication in the θ domain and use the inverse FFT (IFFT) to determine $\chi(\tau, f)$.

We do this for the values $f = f_k$ of interest. Thus, every FFT-multiply-IFFT will result in a range cut of the ambiguity function at $f = f_k$. This is why we say that we develop the ambiguity function as a sequence of range cuts.

For the second technique, we construct the ambiguity function as a sequence of Doppler cuts. In this case, note that, for some $\tau = \tau_k$, we can interpret $\chi(\tau_k, f)$ as the inverse Fourier transform of $u(t)v^*(t + \tau_k)$. In this case we compute $u(t)v^*(t + \tau_k)$ for various $\tau = \tau_k$ and use the IFFT to approximate the appropriate inverse Fourier transform. Each of these is a Doppler cut at mismatch range τ_k . The aggregation of these is the ambiguity function.

4.5 AMBIGUITY FUNCTION GENERATION USING THE FFT

We have already discussed that we can use Fourier transforms to compute the ambiguity function by computing individual range cuts. Specifically, let $u(t)$ be the baseband transmitted signal and $v(t)$ be the signal to which the matched filter (or signal processor) is matched. Let f be the Doppler mismatch at which we want to generate a range cut. The algorithm used to generate the range cut, $|\chi(\tau, f)|$, is

- Find $\mathfrak{T}[u(t)e^{j2\pi ft}] = F_u(\theta)$
- Find $\mathfrak{T}[v(t)] = F_v(\theta)$
- Find $F_\chi(\theta) = F_u(\theta)F_v^*(\theta) = \mathfrak{T}[\chi(\tau, f)]$
- Find $\chi(\tau, f) = \mathfrak{T}^{-1}[F_\chi(\theta)]$

In the above, we can use the FFT and inverse FFT to approximate $\mathfrak{T}[\bullet]$ and $\mathfrak{T}^{-1}[\bullet]$.

When using the FFT we need to be sure we satisfy the sampling theorem and account for some time properties of the FFT and ambiguity function. The algorithm is as follows

1. Create a time array, t , whose length, N , is an integer power of 2 (i.e., $N = 2^M$ where M is a positive integer) and extends from 0 to T , where T is equal to, or greater than, the sum of the durations of $u(t)$ and $v(t)$. This assures that the resulting range cut will cover all time values over which the range cut is not zero. Some restrictions on N are:
 - Choose N so that N/T satisfies Shannon's sampling theorem. That is choose N such that $N/T > 2F$ where F is the highest frequency of $u(t)$ and/or $v(t)$.
 - The larger you choose N the better the range cut will look.
2. Compute $v = v(t)$. Note: $v(t)$ will contain several zero values at the end.
3. Compute $\text{FFT}(v) = F_v$.
4. Select a Doppler mismatch frequency, f .
5. Compute $u = u(t)e^{j2\pi ft}$.
6. Compute $\text{FFT}(u) = F_u$.
7. Compute $|\chi(\tau', f)| = X = |\text{IFFT}(F_u F_v^*)|$.
 - Because of the operation of the FFT and IFFT, $\tau' = 0$ corresponds to the 1st tap, $\tau' = T/N$ corresponds to the 2nd tap, $\tau' = -T/N$ corresponds to the N^{th} FFT tap, and so forth.
 - To get the $\tau' = 0$ point in the center of a plot of $|\chi(\tau', f)|$ vs. τ' , which is desired, we need to rearrange the FFT outputs and redefine a time array
8. Swap the upper and lower $N/2$ samples of X . In MATLAB this is accomplished by using the `fftshift` function.
9. Create a τ array that contains N samples and extends from $-T/2$ to $T/2 - T/N$.
 - When you plot X vs. τ you have a range cut of the ambiguity function at a mismatch Doppler of f .
10. To create another range cut at a different Doppler mismatch, repeat steps 4 through 9.
11. To create a 3-dimensional-looking depiction of the ambiguity function (as in *Figure 4-7*), assemble the series of range cuts and plot them using a 3-D plotting routine such as the `surf` plotting function in MATLAB.