3.7 MATCHED FILTER

3.7.1 Problem Statement

In the detection probability equations ((3-106), (3-109) and (3-110)) we note that \( P_d \) depends directly on SNR. That is, \( P_d \) increases as SNR increases. Because of this we want to try to be sure that the receiver is designed to maximize SNR. We do this by including a matched filter in the receiver. As indicated in Figures 3-1 and 3-2 the matched filter is usually included right before the signal processor. In fact, in some radars, the matched filter is the signal processor.

The statements in the above paragraph provide the design requirement for the matched filter. Specifically, given some signal, \( s(t) \), and noise \( n(t) \), we want to find a filter impulse response, \( h(t) \), that maximizes SNR at the filter output. For purposes of this discussion we assume that the signal is not a random process. Actually, what we are assuming is that the form of the signal is deterministic (its amplitude can still be random). This is the case in practice, even for SW2 or SW4 targets.

As indicated in Figure 3-8 if the input to the matched filter is \( s(t) \) the output will be \( s_o(t) \). If the input is \( n(t) \) the output will be \( n_o(t) \). The output, instantaneous, normalized, signal power is

\[
P_{s_o}(t) = |s_o(t)|^2.
\]

For purposes of the matched filter design, we define the peak signal power at the matched filter output as

\[
P_S = \max_i P_{s_o}(t) = P_{s_o}(t_o) = |s_o(t_o)|^2.
\]

Since \( n_o(t) \) is a random process (that we assume is wide-sense stationary) we must work with its average power. Thus, the average noise power at the output of the matched filter is

\[
P_n = E\left(|n_o(t)|^2\right).
\]

With the above we can now define the design criterion for the matched filter. Specifically, we choose the matched filter so as to maximize the ratio of peak signal power to average noise power at the output of the matched filter. In equation form,

\[
h(t): \max_{h(t)} \frac{P_S}{P_n}
\]

\[\text{6} \text{ The choice of symbols for peak signal and average noise power is not coincidental. As will be shown, these are the same peak signal and average noise power terms included in the radar range equation.}\]
3.7.2 Problem Solution

Equation (3-124) states that we must first write \( P_s \) and \( P_n \) in terms of \( h(t) \) and then maximize it with respect to \( h(t) \).

If we assume that \( h(t) \) is linear we can write
\[
s_o(t) = s(t) * h(t)
\]
and
\[
n_o(t) = n(t) * h(t)
\]
where * denotes convolution.

We will choose to solve the optimization problem in the frequency domain through the use of Fourier transforms. To this end we write
\[
H(f) = \mathfrak{F}[h(t)] ,
\]
\[
S(f) = \mathfrak{F}[s(t)] \quad \text{and}
\]
\[
S_o(f) = \mathfrak{F}[s_o(t)]
\]
and
\[
S_o(f) = H(f)S(f).
\]
In the above \( \mathfrak{F}[x] \) denotes the Fourier transform.

Since \( n(t) \) and \( n_o(t) \) are random processes we must deal with them as such.

This means we write
\[
N(f) = \mathfrak{F}\left[E\left[n(t + \tau)\right]\right] ,
\]
\[
N_o(f) = \mathfrak{F}\left[E\left[n_o(t + \tau)\right]\right] \quad \text{and}
\]
\[
N_o(f) = |H(f)|^2 N(f).
\]
Although not obvious we took advantage of the fact that \( n(t) \) and \( n_o(t) \) are wide-sense stationary.
We now recognize that \( N_o(f) \) is a power spectral density and thus that
\[
P_n = \int_{-\infty}^{\infty} N_0(f) df = \int_{-\infty}^{\infty} |H(f)|^2 N(f) df.
\] (3-134)
The peak signal power is given by
\[
P_S = |s_o(t_o)|^2.
\] (3-135)
However, we can write
\[
s_o(t_o) = \mathcal{F}^{-1} \left[ S_o(f) \right] = \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi f t_o} df.
\] (3-136)
If we combine Equations (3-136), (3-135), (3-134) and (3-124) we get
\[
h(t) : \max_{h(t)} \frac{P_S}{P_n} = \max_{h(t)} \frac{\int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi f t_o} df}{\int_{-\infty}^{\infty} |H(f)|^2 N(f) df}.
\] (3-137)
At this point we make the assumption that \( n(t) \) is white with a noise power spectral density of
\[
N(f) = kT_n F_n G
\] (3-138)
and write
\[
h(t) : \max_{h(t)} \frac{\left| \int_{-\infty}^{\infty} S(f) H(f) e^{j2\pi f t_o} df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 df}.
\] (3-139)
In the above equation \( G \) is the gain of all receiver components between the antenna and the input to the matched filter. Thus, \( N(f) \) is the noise power spectral density at the input to the matched filter.

We start the maximization process by applying one of the Cauchy-Schwartz inequalities to the numerator. The particular Cauchy-Schwartz inequality of interest is
\[
\left| \int_{a}^{b} A(f) B(f) df \right|^2 \leq \left( \int_{a}^{b} |A(f)|^2 df \right) \left( \int_{a}^{b} |B(f)|^2 df \right)
\] (3-140)
with equality when
\[
A(f) = KB^*(f)
\] (3-141)
where $K$ is an arbitrary (complex) constant. If we apply Equation (3-140) to the ratio of Equation (3-139) with the associations

$$A(f) = H(f)$$

and

$$B(f) = S(f)e^{j2\pi f_0}$$

we get

$$\left| \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi f_0} df \right|^2 \leq \left( \int_{-\infty}^{\infty} |H(f)|^2 df \right) \left( \int_{-\infty}^{\infty} |S(f)|^2 df \right)$$

where we have made use of the fact that $|S(f)e^{j2\pi f_0}| = |S(f)|$. (3-145)

We note that Equation (3-144) reduces to

$$\left| \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi f_0} df \right|^2 \leq \frac{\int_{-\infty}^{\infty} |S(f)|^2 df}{kT_0F_nG \int_{-\infty}^{\infty} |H(f)|^2 df}. \quad (3-146)$$

Equation (3-146) tells us that for all $H(f)$ the upper bound on the left side is equal to the right side. That is, we have found the maximum value of $P_s/P_n$ over all $h(t)$, and have solved part of the maximization problem. To find the $h(t)$ that yields the maximum $P_s/P_n$ we invoke the second part of the Cauchy-Schwartz inequality given in Equation (3-141). Specifically, we say that

$$\max_{h(t)} \frac{P_s}{P_n} = \frac{\int_{-\infty}^{\infty} |S(f)|^2 df}{kT_0F_nG}$$

when we choose $H(f)$ as

$$H(f) = KS^*(f)e^{-j2\pi f_0}. \quad (3-148)$$

Thus, we have found the Fourier transform of the filter impulse response that maximizes peak signal to average noise power at the filter output. Furthermore, we have an equation for the maximum in the form of Equation (3-147) and have determined that the maximum occurs at $t = t_0$. (3-147)

We note from the form of Equation (3-148) that
\[ |H(f)| = |KS(f)|. \]  \hspace{1cm} (3-149)

In other words, the matched filter frequency response has the same shape as the frequency spectrum of the signal. They simply differ by a scaling factor \(|K|\). This is the reason we term \(H(f)\) a matched filter.

We now want to look at the specific form of \(h(t)\) relative to \(s(t)\). We can write

\[
h(t) = S^{-1}\left[KS^*(f)e^{-j2\pi ft_o}\right] = \int_{-\infty}^{\infty} KS^*(f)e^{-j2\pi ft_o}e^{j2\pi ft} df
\]

\[
= K \int_{-\infty}^{\infty} S^*(f)e^{-j2\pi f(t-o-t)} df = K \left[ \int_{-\infty}^{\infty} S(f)e^{j2\pi f(t-o-t)} df \right] = Ks^*(t_o-t).
\]  \hspace{1cm} (3-150)

Thus, \(h(t)\) is the conjugate of a scaled (by \(K\)), time reversed (because of the \(-t\)) and shifted (by \(t_o\)) version of the transmit signal, \(s(t)\). This is another reason we refer to \(h(t)\) as a matched filter. This operation is illustrated in Figure 3-9. The left sketch of this figure is that of \(s(t)\) while the right figure is a sketch of \(s^*(-t)\) (we have assumed \(K=1\) since it is arbitrary). Finally, the right figure is \(s^*(t_o-t)\).

![Figure 3-9 – Evolution of \(h(t)\)](image)

Now that we have established the equation for the maximum value of the SNR at the output of the matched filter, and have a filter that can provide the maximum SNR, we want to determine its value. Specifically, we want to relate the maximum SNR to values of SNR we compute from the radar range equation.

We start by recognizing that \(s(t)\) is the radar pulse at the input to the matched filter. It is the rectangular pulse that we have been discussing in our previous work. Thus, we can write \(s(t)\) as
\[ s(t) = V_{\text{rect}} \left[ \frac{t - \tau_R}{\tau_p} \right] \] (3-151)

where \( \tau_R \) is the range delay to the target and \( \tau_p \) is the pulse width. The representation of Equation (3-151) is idealized in that it assumes that the receiver components prior to the matched filter can pass a rectangular pulse without distortion. Although this is not realistic, for purposes of this development it is a fairly reasonable assumption. \( V \) is, in general, a complex number that may include a carrier term (the IF frequency), a phase and, possibly, some type of phase modulation.

When we derived the radar range equation we concluded that the signal power at the output of the antenna was

\[ P_s = \frac{P_t G_r G_p \lambda^2 \sigma}{(4\pi)^3 R^4 L}. \] (3-152)

From the discussion below Equation (3-139), the gain of the receiver from the antenna to the input to the matched filter is \( G \). With this the signal the power at the input to the matched filter is

\[ P_M = P_s G = \frac{P_t G_r G_p \lambda^2 \sigma G}{(4\pi)^3 R^4 L}. \] (3-153)

If we assume a normalized impedance of 1 ohm, we get

\[ |V| = \sqrt{P_M} = \sqrt{\frac{P_t G_r G_p \lambda^2 \sigma G}{(4\pi)^3 R^4 L}}. \] (3-154)

From Equation (3-147) we have

\[ \text{SNR}_{\text{MAX}} = \frac{\int_{-\infty}^{\infty} |S(f)|^2 df}{k T_0 F_n G}. \] (3-155)

Since \( s(t) \) has finite energy and power we can invoke Parseval’s Theorem for the numerator and write

\[ \text{SNR}_{\text{MAX}} = \frac{\int_{-\infty}^{\infty} \left| s(t) \right|^2 dt}{k T_0 F_n G}. \] (3-156)

Substituting Equation (3-151) yields
\[ SNR_{\text{MAX}} = \frac{\int_{-\infty}^{\infty} |V|^2 \text{rect} \left( \frac{t - \tau_R}{\tau_p} \right) dt}{kT_0 F_n G} = \frac{|V|^2}{kT_0 F_n G} \]  

(3-157)

or with Equation (3-154)

\[ SNR_{\text{MAX}} = \frac{P_t G_T G_R \lambda^2 \sigma G \tau_p}{(4\pi)^3 R^4 k T_0 F_n G} = \frac{P_t G_T G_R \lambda^2 \sigma G \tau_p}{(4\pi)^3 R^4 k T_0 F_n L}. \]  

(3-158)

We recognize Equation (3-158) as the SNR that we found when we derived the radar range equation. This tells us that the peak value of SNR at the output of the matched filter is the SNR we obtain from the radar range equation. In essence, it says that the matched filter ekes out the maximum possible SNR from the signal and noise that the radar must deal with. There is no other filter that will give a larger value of SNR for the transmitted signal and the case where the interference is due to white noise at the input to the matched filter. If the interference is other than white noise, e.g. clutter, there are other filters that will provide larger values of signal-to-interference power ratio (SIR) than the filter defined by Equation (3-150).

3.7.3 Matched Filter Response for an Unmodulated Pulse

3.7.3.1 General Formulation

We want to derive an equation for the matched filter response for a signal, \( s(t) \), and then use it to derive the matched filter response for the case where \( s(t) \) is an unmodulated pulse. The results can also be extended to finding the matched filter response when \( s(t) \) is an LFM pulse.

From Equation (3-150) we have

\[ h(t) = K s^\ast(t_0 - t) \]  

(3-159)

where \( K \) is an arbitrary (complex) constant and \( t_0 \) is the value of \( t \) at which the matched filter response to \( s(t) \) will reach its peak.

Since we can pick \( K \) and \( t_0 \) to be anything we want, we can let, without loss of generality, \( K = 1 \) and \( t_0 = 0 \). The latter statement says that the output of the matched filter will reach its peak at a relative time of zero. With this we get

\[ h(t) = s^\ast(-t). \]  

(3-160)

The response of \( h(t) \) to \( s(t) \) is given by

\[ s_o(t) = h(t) \ast s(t) = \int_{-\infty}^{\infty} s(\alpha) h(t - \alpha) d\alpha. \]  

(3-161)
But \( h(t) = s^\ast(-t) \) so \( h(t-\alpha) = s^\ast(-(t-\alpha)) = s^\ast(\alpha-t) \) and

\[
s_v(t) = \int_{-\infty}^{\infty} s(\alpha) s^\ast(\alpha-t) \, d\alpha.
\]  (3-162)

### 3.7.3.2 Response for an Unmodulated Pulse

For purposes of the rest of the analyses we will treat \( s(t) \) and \( s^\ast(t) \) as separate functions. Let

\[
s(t) = Ae^{j\theta} \text{rect} \left[ \frac{t - \tau_p/2}{\tau_p} \right]
\]  (3-163)

and

\[
s^\ast(t) = Ae^{-j\theta} \text{rect} \left[ \frac{t - \tau_p/2}{\tau_p} \right].
\]  (3-164)

A plot of \( s(t) \) is shown in Figure 3-10. The plot of \( s^\ast(t) \) would look the same except that the “height” would be \( Ae^{-j\theta} \) rather than \( Ae^{j\theta} \).

![Figure 3-10 – Unmodulated Pulse](image)

In the \( s_v(t) \) integral we note that \( t \) is the separation between \( s(\alpha) \) and \( s^\ast(\alpha-t) \) as shown in Figure 3-11. Figure 3-11 corresponds to the case where \( t \geq 0 \).

![Figure 3-11 – Plot of \( s(\alpha) \) and \( s^\ast(\alpha-t) \) for \( t \geq 0 \)](image)
It should be clear that if \( t \geq \tau_p \) or \( t \leq -\tau_p \), or \( |t| \geq \tau_p \), then \( s(\alpha) \) and \( s^*(\alpha-t) \) will not overlap, \( s(\alpha)s^*(\alpha-t)=0 \) and \( s_o(t)=0 \). Thus
\[
s_o(t)=0 \quad |t| \geq \tau_p. \tag{3-165}
\]

For \( 0 \leq t < \tau_p \) the overlap region of \( s(\alpha) \) and \( s^*(\alpha-t) \) is \( t \leq \alpha < \tau_p \). Further, over this region \( s(\alpha)s^*(\alpha-t)=Ae^{i\theta}Ae^{-i\theta}=A^2 \) and thus
\[
s_o(t)=\int_{t}^{\tau_p} A^2 \, d\alpha = A^2 \left( \tau_p - t \right). \tag{3-166}
\]

We note that since \( t \geq 0 \), \( |t|=t \). Thus we replace \( t \) with \( |t| \) to get
\[
s_o(t)=A^2 \left( \tau - |t| \right) \quad 0 \leq t < \tau_p. \tag{3-167}
\]

The arrangement of \( s(\alpha) \) and \( s^*(\alpha-t) \) for \( -\tau_p < t < 0 \) is shown in Figure 3-12.

In this case the overlap region is \( 0 < \alpha < t + \tau_p \) and
\[
s_o(t)=\int_{0}^{t+\tau_p} A^2 \, d\alpha = A^2 \left( \tau_p + t \right). \tag{3-168}
\]

Here we note that since \( t < 0 \), \( t = -|t| \). Thus we replace \( t \) with \(-|t|\) to get
\[
s_o(t)=A^2 \left( \tau_p - |t| \right) \quad \tau_p < t < 0. \tag{3-169}
\]

Since this is the same form as for \( 0 \leq t < \tau_p \), we can combine these to get
\[
s_o(t)=A^2 \left( \tau_p - |t| \right) \quad |t| < \tau_p. \tag{3-170}
\]

Finally, if we combine this with the result for \( |t| \geq \tau_p \), we get
\[
 s_o(t) = \begin{cases} 
 0 & |t| \geq \tau_p \\
 A^2 \left( \tau_p - |t| \right) & |t| < \tau_p 
\end{cases} = A^2 \left( \tau_p - |t| \right) \text{rect} \left[ \frac{t}{2\tau_p} \right].
\]

(3-171)

A plot of \( s_o(t) \) is shown in Figure 3-13.

![Figure 3-13 – Plot of Matched Filter Output](image)