4.0 COVARIANCE PROPAGATION

4.1 Introduction

Now that we have completed our review of linear systems and random processes, we want to examine the performance of linear systems excited by random processes. The standard approach used to analyze the outputs of linear systems excited by random processes is based on the power spectrum and Fourier transforms, Laplace transforms (for continuous-time systems and random processes) and Z-transforms (for discrete-time systems and random processes). A basic assumption of this method is that the input random process is at least wide-sense stationary and that we are examining the output of the system after it reaches steady state, so that the output of the system is also stationary (at least wide-sense stationary). The method also makes the assumption that the linear system is not time varying.

In Kalman filter theory we will deal with non-stationary input processes, and we will be interested in the transient behavior of the output and the linear systems will be time varying. Because of this we need to introduce an analysis methodology that will accommodate these properties of the random processes and system. This method is termed covariance propagation. It is a time-domain analysis technique that allows us to derive straightforward equations for the mean of the system state, the covariance matrix of the system state and the autocovariance matrix of the system state. From this we can also formulate equations for the mean, covariance matrix and autocovariance matrix of the output. We could also develop equations for the correlation matrix and the autocorrelation matrix. We omit the derivation here because these matrices aren’t used much in practice.

We will derive the covariance propagation methodology for continuous- and discrete-time systems. We present both derivations because they are different. It should be noted that the concept of covariance propagation and the derivations of this chapter are not critical to the derivation and understanding of Kalman filters. However, the derivations are interesting and the concept of covariance propagation is not normally taught in other courses on random processes or linear system theory.

4.2 Continuous-Time Systems and Processes

4.2.1 Problem Definition

Let the state variable representation of a linear system be

\[ \mathbf{x}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \]  

(4-1)

where \( \mathbf{x}(t) \) is the state vector and \( \mathbf{u}(t) \) is the input vector. In this case we assume that the input is a random process. This means that the state will also be a random process. The matrices \( A(t) \) and \( B(t) \) are the system and input distribution matrices, respectively. The solution to Equation 4-1 is given by (see Equations 2-16 through 2-18)
\[ x(t) = \Phi(t,t_0) x(t_0) + \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau) d\tau \]  

(4-2)

where \( \Phi(t,t_0) \) is the state transition matrix and satisfies the matrix differential equation

\[ \Phi(t,t_0) = A(t) \Phi(t,t_0) \]  

(4-3)

with

\[ \Phi(t_0,t_0) = I . \]  

(4-4)

As indicated above, since \( u(t) \) is a random process \( x(t) \) is also a random process. Because \( x(t) \) is a random process, information about it cannot be easily extracted from samples of \( x(t) \). To gain useful information about \( x(t) \) we would ideally like to derive the joint density function of its elements. This, however, is very difficult, if not impossible, to derive. Thus, we use the mean and covariance matrix to characterize \( x(t) \).

4.2.2 Mean

We can take the expected value of \( Equation \ 4-2 \) as

\[ E\{x(t)\} = E\{\Phi(t,t_0) x(t_0)\} + E\left\{ \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau) d\tau \right\} . \]  

(4-5)

Taking note of the fact that \( \Phi(t,t_0) \) and \( B(t) \) are not random processes and making use of the fact that the expected value of an integral is the integral of the expected value, we obtain

\[ E\{x(t)\} = \Phi(t,t_0) E\{x(t_0)\} + \int_{t_0}^{t} \Phi(t,\tau) B(\tau) E\{u(\tau)\} d\tau \]  

(4-6)

or, letting \( \eta_u(t) = E\{u(t)\} \) and \( \eta_x(t) = E\{x(t)\} \) we obtain

\[ \eta_x(t) = \Phi(t,t_0) \eta_x(t_0) + \int_{t_0}^{t} \Phi(t,\tau) B(\tau) \eta_u(\tau) d\tau . \]  

(4-7)

Thus, we are able to express the mean, \( \eta_x(t) \), of the state, \( x(t) \), as a function of the mean, \( \eta_u(t) \), of the random input, \( u(t) \).

While \( Equation \ 4-7 \) is an interesting equation, it is of little use in practical calculations of \( \eta_x(t) \) from \( \eta_u(t) \). This stems from the fact that \( Equation \ 4-7 \) requires the (complicated) computation of \( \Phi(t,t_0) \) and the integral. To convert this equation to a form that is more easily implemented, we make use of some of our state variable theory. If we differentiate both sides of \( Equation \ 4-7 \) with respect to \( t \), we obtain
\[ \dot{\eta}_s(t) = \frac{d}{dt} \left[ \Phi(t, t_0) \eta_s(t_0) \right] + \frac{d}{dt} \left[ \int_{t_0}^{t} \Phi(t, \tau) B(\tau) \eta_u(\tau) d\tau \right] \]  
(4-8)

or

\[ \dot{\eta}_s(t) = \Phi(t, t_0) \eta_s(t_0) + \Phi(t, t) B(t) \eta_u(t) + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) \eta_u(\tau) d\tau \]  
(4-9)

where the last two terms are a result of employing Leibniz’s rule. Making use of Equations 4-3 and 4-4 and grouping terms results in

\[ \dot{\eta}_s(t) = A(t) \Phi(t, t_0) \eta_s(t_0) + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) \eta_u(\tau) d\tau + B(t) \eta_u(t) \]  
(4-10)

Referring to Equation 4-7, we recognize the bracketed term in Equation 4-10 as being \( \eta_s(t) \). Thus, we can rewrite Equation 4-10 as

\[ \dot{\eta}_s(t) = A(t) \eta_s(t) + B(t) \eta_u(t) \]  
(4-11)

Equation 4-11 tells us that the differential equation for the mean of the state is of the same form as the differential equation of the state. The driving function, or input, is the mean of the original driving function. Practically, the form of Equation 4-11 is very attractive since we can use numerical integration to solve the differential equation for the mean of the state.

4.2.3 Covariance

Next we turn our attention to determining the covariance matrix of the state. That is we want to find an equation for

\[ P(t) = E \left\{ [x(t) - \eta_s(t)][x(t) - \eta_s(t)]^T \right\} \]  
(4-12)

Based on Equation 4-11 we will attempt to find a differential equation for \( P(t) \).

The first step in our derivation is to obtain an expression for \( x(t) - \eta_s(t) \). To do so we subtract Equation 4-7 from Equation 4-2 to yield

\[ x(t) - \eta_s(t) = \Phi(t, t_0)[x(t_0) - \eta_s(t_0)] + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) [u(\tau) - \eta_u(\tau)] d\tau \]  
(4-13)

or, with \( \Delta x(t) = x(t) - \eta_s(t) \) and \( \Delta u(\tau) = u(\tau) - \eta_u(\tau) \),

\[ \Delta x(t) = \Phi(t, t_0) \Delta x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) \Delta u(\tau) d\tau \]  
(4-14)

Next we form \( P(t) = E \left\{ [\Delta x(t)][\Delta x(t)]^T \right\} \) to yield
\[ P(t) = E \left[ \Delta x(t) \Delta x^T(t) \right] = E \{ \Phi(t,t_0) \Delta x(t_0) \Delta x^T(t_0) \Phi^T(t,t_0) \} \]

\[ \quad + \int_{t_0}^t \Phi(t,\tau)B(\tau)\Delta u(\tau)\Delta x^T(t_0) d\tau \Phi^T(t,t_0) \]

\[ \quad + \Phi(t,t_0) \int_{t_0}^t \Delta x(t_0) \Delta u^T(\gamma) B^T(\gamma) \Phi^T(t,\gamma) d\gamma \]

\[ \quad + \int_{t_0}^t \int_{t_0}^t \Phi(t,\tau) B(\tau) C_u(\tau,\gamma) B^T(\gamma) \Phi^T(t,\gamma) d\gamma d\tau \] \quad \quad (4-15)

For the next step we note that the expected value of a sum is the sum of expected values and the expected value of an integral is the integral of the expected value. We also make use of the definitions

\[ P_{ux}(t,t_0) = E \{ \Delta u(t) \Delta x^T(t_0) \} \]

\[ P_{ux}(t_0,t) = E \{ \Delta x(t_0) \Delta u^T(t) \} \]

\[ C_u(t,\tau) = E \{ \Delta u(t) \Delta u^T(\tau) \} \]

With this we get

\[ P(t) = \Phi(t,t_0) P(t_0) \Phi^T(t,t_0) \]

\[ \quad + \int_{t_0}^t \Phi(t,\tau)B(\tau)P_{ux}(\tau,t_0) d\tau \Phi^T(t,t_0) \]

\[ \quad + \Phi(t,t_0) \int_{t_0}^t P_{ux}(t_0,\gamma) B^T(\gamma) \Phi^T(t,\gamma) d\gamma \]

\[ \quad + \int_{t_0}^t \int_{t_0}^t \Phi(t,\tau) B(\tau) C_u(\tau,\gamma) B^T(\gamma) \Phi^T(t,\gamma) d\gamma d\tau \] \quad \quad (4-16)

\[ \text{Equation 4-16 tells us that if we know, or can compute, the required terms, we have an expression for the covariance of the state. Again, finding the matrices } \Phi(t,t_0) \text{ and evaluating the required integrals is, generally, extremely difficult, if not impossible. Thus, as in the case of the mean, we seek a simpler form. First, we observe that it is very reasonable to assume that the input random process will be independent of, or at least uncorrelated with, the initial state, } x(t_0). \text{ With this } P_{ux}(t,t_0) = 0 \text{ and } P_{ux}(t_0,t) = 0, \text{ where } 0 \text{ is the null matrix. This allows us to reduce Equation 4-16 to} \]

\[ P(t) = \Phi(t,t_0) P(t_0) \Phi^T(t,t_0) + \int_{t_0}^t \int_{t_0}^t \Phi(t,\tau) B(\tau) C_u(\tau,\gamma) B^T(\gamma) \Phi^T(t,\gamma) d\gamma d\tau . \quad (4-17) \]

To reduce Equation 4-17 further we assume that the input random process is white. This yields

\[ C_u(\tau,\gamma) = P_u(\tau) \delta(\tau - \gamma) \quad \quad (4-18) \]

and

\[ P(t) = \Phi(t,t_0) P(t_0) \Phi^T(t,t_0) + \int_{t_0}^t \Phi(t,\tau) B(\tau) P_u(\tau) B^T(\tau) \Phi^T(t,\tau) d\tau . \quad (4-19) \]
While *Equation 4-19* is considerably simpler than *Equation 4-16*, its evaluation is very difficult. As we did for the mean, we will take the derivative of *Equation 4-19* in hopes of deriving a differential equation for $P(t)$. Proceeding we get

$$
\dot{P}(t) = \Phi(t,t_0) P(t_0) \Phi^T(t,t_0) + \Phi(t,t_0) P(t_0) \left( \Phi(t,t_0) \right)^T \\
+ \Phi(t,t) B(t) P_u(t) B^T(t) \Phi^T(t,t) \\
+ \int_{t_0}^t \Phi(t,\tau) B(\tau) P_u(\tau) B^T(\tau) \Phi^T(t,\tau) d\tau \\
+ \int_{t_0}^t \Phi(t,\tau) B(\tau) P_u(\tau) B^T(\tau) \left( \Phi(t,\tau) \right)^T d\tau
$$

*(4-20)*

Making use of *Equation 4-3*, *4-4* and *4-19*, and grouping terms, we get

$$
\dot{P}(t) = P(t) A^T(t) + A(t) P(t) + B(t) P_u(t) B^T(t).
$$

*(4-21)*

Note that *Equation 4-21* is a matrix differential equation for $P(t)$ in terms of the (known) system and input distribution matrices, $A(t)$ and $B(t)$ respectively, and the spectral density amplitude, $P_u(t)$, of the input random process. As with the mean, the equation for $P(t)$ is very easy to solve using numerical integration techniques.

### 4.2.4 Summary

In summary, for a continuous-time system represented by the state variable equation

$$
\dot{x}(t) = A(t) x(t) + B(t) u(t)
$$

*(4-22)*

where $u(t)$ is a random process with mean $\eta_u(t)$, the mean of $x(t)$, $\eta_x(t)$, is defined by the vector differential equation

$$
\dot{\eta}_x(t) = A(t) \eta_x(t) + B(t) \eta_u(t).
$$

*(4-23)*

Furthermore, if $u(t)$ is white and uncorrelated with the initial state, $x(t_0)$, and has a noise power spectral density amplitude $P_u(t)$, then the covariance of the state, $P(t)$, is defined by the matrix differential equation

$$
\dot{P}(t) = P(t) A^T(t) + A(t) P(t) + B(t) P_u(t) B^T(t).
$$

*(4-24)*

*Equation 4-24* is a form of the Matrix Riccatti equation that occurs in several developments related to optimal control theory and state estimation. We will see this equation again when we derive the Kalman filter for continuous time systems.

Two interesting extensions of *Equation 4-24* are posed as homework problems. One of these is to extend the development to the case where the
input random process is colored and the other is to derive an equation for the autocovariance matrix,

\[
C(t+\tau,t) = E\left\{\left[x(t+\tau) - \eta_x(t+\tau)\right]\left[x(t) - \eta_x(t)\right]^T\right\}
\]  

(4-25)
in terms of the covariance matrix, \( P(t) \).

### 4.3 Discrete-Time Systems and Processes

#### 4.3.1 Problem Definition

We now consider discrete-time systems of the form

\[
x(k+1) = F(k)x(k) + G(k)u(k)
\]  

(4-26)

where \( x(k) \) is the state vector, \( u(k) \) is the input random process, \( F(k) \) is the system matrix and \( G(k) \) is the input distribution matrix. As with continuous-time systems, we seek a characterization of the state in terms of its mean and covariance.

#### 4.3.2 Mean

Taking the expected value of \( \text{Equation 4-26} \) we get

\[
E\{x(k+1)\} = E\{F(k)x(k) + G(k)u(k)\}
\]

\[
= F(k)E\{x(k)\} + G(k)E\{u(k)\}.
\]  

(4-27)

Denoting \( E\{x(k)\} \) as \( \eta_x(k) \) and \( E\{u(k)\} \) as \( \eta_u(k) \) we can rewrite \( \text{Equation 4-27} \) as

\[
\eta_x(k+1) = F(k)\eta_x(k) + G(k)\eta_u(k).
\]  

(4-28)

\( \text{Equation 4-28} \) tells us that, like continuous-time systems, the mean of the state is described by the same state variable equation as the state itself.

#### 4.3.3 Covariance

We next want to find an equation for the covariance of the state. To do this we use essentially the same approach that we used for continuous-time systems. Specifically we define \( \Delta x(k) = x(k) - \eta_x(k) \) and \( \Delta u(k) = u(k) - \eta_u(k) \) and subtract \( \text{Equation 4-28} \) from \( \text{Equation 4-26} \) to yield

\[
\Delta x(k+1) = F(k)\Delta x(k) + G(k)\Delta u(k).
\]  

(4-29)

We next form \( P(k) = E\{[\Delta x(k)][\Delta x(k)]^T\} \) and use the definitions
\[ P_{ux} (k) = E \{ \Delta u (k) \Delta x^T (k) \} \]
\[ P_{uu} (k) = E \{ \Delta x (k) \Delta u^T (k) \} \]
\[ P_u (k) = E \{ \Delta u (k) \Delta u^T (k) \} \]
to write
\[ P(k+1) = F (k) P(k) F^T (k) + G(k) P_u (k) G^T (k) + F (k) P_{ux} (k) G^T (k) + G(k) P_{ux} (k) F^T (k) . \]  
(4-30)

To further simplify Equation 4-30 we will show that \( P_{ux} (k) = 0 \) and \( P_{uu} (k) = 0 \) when \( u (k) \) is white and uncorrelated with the initial state. To do this we must first digress to derive a representation of Equation 4-29 that explicitly includes the initial state and all input vectors up to stage \( k \). We will pose this representation in terms of a theorem and then prove the theorem by induction.

**THEOREM 4-1:** Given the system \( \Delta x (k+1) = F (k) \Delta x (k) + G (k) \Delta u (k) \) and an initial state, \( \Delta x (0) \), an equivalent representation for \( \Delta x (k+1) \) is

\[ \Delta x (k+1) = F_{0,k} \Delta x (0) + \sum_{m=0}^{k-1} F_{m+1,k} G(m) \Delta u (m) + G(k) \Delta u (k) \]  
(4-31)

where \( F_{m,k} = \prod_{n=m}^{k} F (n) \).

In this theorem, we have arbitrarily taken the initial stage to be \( k = 0 \). This represents no loss of generality when compared to letting the initial stage be \( k = k_0 \).

As we indicated above, we will prove this theorem by induction. First, we show that the theorem is true for a specific value of \( k \). We will choose \( k = 1 \). We note that, for \( k = 0 \),
\[ \Delta x (1) = F (0) \Delta x (0) + G (0) \Delta u (0) . \]  
(4-32)

For \( k = 1 \) we get
\[ \Delta x (2) = F (1) \Delta x (1) + G (1) \Delta u (1) . \]  
(4-33)
Combining Equations 4-32 and 4-33, we get
\[ \Delta x (2) = F (1) F (0) \Delta x (0) + F (1) G (0) \Delta u (0) + G (1) \Delta u (1) \]  
(4-34)
which is the same form as Equation 4-31. Thus, the theorem is proved for \( k = 1 \).

For the second part of the induction proof, we assume that the theorem is true for some arbitrary \( k = n - 1 \). Thus, we assume that the equation
\[ \Delta x(n) = F_{0,n-1} \Delta x(0) + \sum_{m=0}^{n-2} F_{m+1,n-1} G(m) \Delta u(m) + G(n-1) \Delta u(n-1) \] (4-35)

is correct.

For the third part of the induction proof, we use the results of part two to show that the theorem is true for \( k = n \). Proceeding, from the given of the theorem we have

\[ \Delta x(n+1) = F(n) \Delta x(n) + G(n) \Delta u(n). \] (4-36)

Combining this with Equation 4-35 we get

\[ \Delta x(n+1) = F(n) \left[ F_{0,n-1} \Delta x(0) + \sum_{m=0}^{n-2} F_{m+1,n-1} G(m) \Delta u(m) + G(n-1) \Delta u(n-1) \right] + G(n) \Delta u(n) \] (4-37)

After multiplying by \( F(n) \) and combining the last two terms in brackets, we get

\[ \Delta x(n+1) = F_{0,n} \Delta x(0) + \sum_{m=0}^{n-1} F_{m+1,n} G(m) \Delta u(m) + G(n) \Delta u(n) \] (4-38)

which is the same form as Equation 4-31. Thus, the theorem is proved.

After this digression, we return to the issue of examining \( P_{xu}(k) \) and \( P_{ux}(k) \). Specifically, we want to show that they are zero under the conditions that the initial state is uncorrelated with the input and that the input random process is white. It will suffice to show that \( P_{xu}(k) \) is zero since \( P_{ux}(k) = P_{xu}^T(k) \).

The statement that the initial state is uncorrelated with the input allows us to write

\[ E \left\{ \Delta x(0) \Delta u^T(k) \right\} = 0 \] (4-39)

and the statement that the input random process is white allows us to write

\[ E \left\{ \Delta u(m) \Delta u^T(k) \right\} = P_u(k) \delta_{m,k}. \] (4-40)

Using Equation 4-31 we can write

\[ P_{xu}(k) = E \left\{ \Delta x(k) \Delta u^T(k) \right\} \]

\[ = F_{0,k-1} E \left\{ \Delta x(0) \Delta u^T(k) \right\} + \sum_{m=0}^{k-2} F_{m+1,k-1} G(m) E \left\{ \Delta u(m) \Delta u^T(k) \right\} \]

\[ + G(k) E \left\{ \Delta u(k-1) \Delta u^T(k) \right\} \] (4-41)

The first term in Equation 4-41 is zero because of Equation 4-39 and the summation and last term are zero because of Equation 4-40. Thus, \( P_{xu}(k) \) is zero.
Since $P_{uu}(k)$ and $P_{ux}(k)$ are zero, Equation 4-30 reduces to

$$P(k+1) = F(k)P(k)F^T(k) + G(k)P_u(k)G^T(k).$$

(4-42).

Equation 4-42 is also a form of the matrix Ricatti equation and will appear again in our development of the discrete-time Kalman filter.

4.3.4 Summary

Summarizing the results of this section we have the following. Given the system

$$x(k+1) = F(k)x(k) + G(k)u(k)$$

(4-43)

where $u(k)$ is a random process with a mean of $\eta_u(k)$, the mean of the state, $x(k)$, is given by

$$\eta_x(k+1) = F(k)\eta_x(k) + G(k)\eta_u(k).$$

(4-44)

In Equation 4-44 it is assumed that $\eta_x(k_0)$ is given and that $k \geq k_0$. In general we let $k_0 = 0$ without loss of generality.

If $u(k)$ is white random process with a covariance matrix $P_u(k)$, and is uncorrelated with $x(k_0)$, then the covariance matrix for $x(k)$ for $k \geq k_0$ is given by

$$P(k+1) = F(k)P(k)F^T(k) + G(k)P_u(k)G^T(k)$$

(4-45)

where it is assumed that $P(k_0)$ is given.

As was done for the continuous-time case, two interesting extensions of Equation 4-45 are posed as homework problems. One of these is to extend the development to the case where the input random process is colored and the other is to derive an equation for the autocovariance matrix,

$$C(k+m,k) = E\left\{[x(k+m) - \eta_x(k+m)][x(k) - \eta_x(k)]^T\right\}. $$

(4-46)

in terms of the covariance matrix, $P(k)$.

4.4 A Discrete-Time Example

To illustrate some of the concepts of the previous section, we will consider a simple example. Specifically, we consider the system defined by

$$x(k+1) = 0.9x(k) + 0.1u(k)$$

(4-47)
with \( x(0)=2 \). \( u(k) \) is a white, Gaussian, wide sense stationary random process with a mean of 1 and a variance of 5. This tells us that \( \eta_u(k)=1 \ \forall k \) and that \( P_u(k)=5 \ \forall k \). Note that \( \eta_u(k) \) and \( P_u(k) \) are scalars in this case since \( u(k) \) is a scalar. The fact that \( u(k) \) is Gaussian tells us that the state, \( x(k) \) is also Gaussian.

From *Equation 4-44*, the mean of the state is given by
\[
\eta_x(k+1) = 0.9\eta_x(k) + 0.1\eta_u(k) \tag{4-48}
\]
with \( \eta_x(0)=2 \) since \( x(0)=2 \).

From *Equation 4-45*, the covariance of the state is given by
\[
P_x(k+1) = (0.9)^2 P_x(k) + (0.1)^2 P_u(k) \tag{4-49}
\]
with \( P_x(0)=0 \) since \( x(0) \) is a given number, and thus not a random variable.

*Figure 4-1* contains plots of \( \eta_x(k) \) and \( P_x(k) \) versus \( k \). The non-stationary nature of \( x(k) \) is evident from the fact that \( \eta_x(k) \) and \( P_x(k) \) are not constant. Recall that the rise time of \( \eta_x(k) \) and \( P_x(k) \) depend upon the value of \( F (=0.9 \text{ in this case}) \). As \( F \) gets closer to zero the rise time decreases (the response becomes faster) and as \( F \) approaches one the rise time increases. If \( F \) becomes greater than one the system becomes unstable. If \( F \) becomes negative (with a magnitude less than one) the mean will oscillate but the covariance will appear the same as for a positive \( F \) of equal magnitude.
Figure 4-2 contains plots of five samples of \( u(k) \) and the corresponding \( x(k) \), along with \( \eta(k) \) and \( \eta(k) \pm 3\sqrt{P(k)} \) for each of them. For the plot of \( x(k) \) in particular, it will be noted that the samples of the process stay mostly within the three-sigma bounds of the mean, as expected. It will also be noted that the plots of the samples of \( x(k) \) are considerably smoother than the samples of \( u(k) \). This is also expected in that the system is a low pass filter that reduces the higher frequency variations of the input random process.

Figure 4-3 contains plots of state autocovariance, \( C(k+m,k) \), (see Equation 4-46) vs. \( m \) for two different values of \( k \). Again the nonstationary nature will be noted by the fact that \( C(k+m,k) \) depends upon both \( k \) and \( m \).
Figure 4.2. Plot of the system input and output

Figure 4.3. Plot of the autocovariance for two values of k
4.5 Problems

1. Extend Equation 4.24 to the case where $u(t)$ is colored. Assume that $u(t)$ is defined by $u'(t) = A_u(t)u(t) + B_u(t)w(t)$ where $w(t)$ is a white noise process with a covariance of $P_w(t)$. Assume $u(0) = 0$. Hint: form an augmented state $z(t) = [x^T(t) u^T(t)]$.

2. Show that the autocovariance matrix for a continuous-time system is defined by the equation $\dot{C}(t+\tau,t) = A(t+\tau)C(t+\tau,t)$ where $\dot{C}(t+\tau,t) = dC(t+\tau,t)/d\tau$, $\tau \geq 0$ and $C(t,t) = P(t)$.

3. Extend Equation 4.45 to the case where $u(k)$ is colored. Assume that $u(k)$ is defined by $u(k+1) = F_u(k)u(k) + G_u(k)w(k)$ where $w(k)$ is a white noise process with a covariance of $P_w(k)$. Assume $u(0) = 0$. Hint: form an augmented state $z(k)$ as $z^T(k) = [x^T(k) u^T(k)]$.

4. Show that the autocovariance matrix for a discrete-time system is defined by the equation $C(k+m+1,k) = F(k+m)C(k+m,k)$ where $m \geq 0$ and $C(k,k) = P(k)$.

5. Given that the input, $u(k)$, and the initial state, $x(0)$, in Equation 4.47 are Gaussian, prove that the state is also Gaussian. Hint: use Equation 4.31.

6. Repeat the discrete-time example of Section 4.3 for the case where $x(k+1) = -0.9x(k) + 0.1u(k)$.

7. Generate plots similar to Figures 4-1, 4-2 and 4-3 for the output of the system of Figure 2-4 with $a = 0.9$, $b = -0.9$ and $c = -0.04$. Let the number of samples be 70 instead of the 50 used in the example of Section 4.3.

8. Generate plots similar to Figures 4-1, 4-2 and 4-3 for the output of the system of Figure 2-4 with $a = -0.9$, $b = 0.9$ and $c = -0.04$. Let the number of samples be 70 instead of the 50 used in the example of Section 4.3.