1.0 ANTENNAS

1.1 INTRODUCTION

In EE 619 we discussed antennas from the view point of antenna aperture, beam width and gain, and how they relate. More specifically, we dealt with the equations

\[ G = \frac{4\pi A_e}{\lambda^2} \]  

and

\[ G = \frac{25,000}{\theta_{AZ} \theta_{EL}} \quad \theta_{AZ}, \theta_{EL} \text{ in degrees} \].

In this course we want to develop equations and techniques for finding antenna radiation and gain patterns. That is, we want to develop equations for \( G(\theta, \phi) \) where \( \theta \) and \( \phi \) are orthogonal angles such as azimuth (AZ) and elevation (EL) or angles relative to a normal to the antenna face (as would be used in phased array antennas). We want to be able to produce plots similar to the one shown in Figure 1-1. We want to use the ability to generate antenna patterns to look at beam width, gain, sidelobes and how these relate to antenna dimensions and other factors.

![3-D plot for circular array with Taylor weighting](image)

Figure 1-1 – Sample Antenna Pattern
We will begin with a simple two-element array antenna to illustrate some of the basic aspects of computing antenna radiation patterns and some of the properties of antennas. We will then progress to linear arrays, and then planar phased arrays.

1.2 TWO ELEMENT ARRAY ANTENNA

Assume we have two isotropic radiators separated by a distance \( d \), as shown in Figure 1-2. In this figure, the arc represents part of a sphere located at a distance of \( r \) relative to the center of the radiators. For these studies, we assume that \( r \gg d \). The sinusoids represent the electric field generated by each radiator.

![Figure 1-2 – Two Element Array Antenna](image)

Since the radiators are isotropic the power of each radiates is uniformly distributed over a sphere at some radius \( r \) (recall the radar range equation). Thus the power over some small area, \( \Delta A \), due to either radiator is given by

\[
P_s = \frac{P \Delta A}{4\pi r^2} = \frac{K^2}{r^2}
\]

(1-3)

Since the electric field intensity at \( r \) is proportional to \( \sqrt{P_s} \) we can write

\[
|E| = \frac{E_s}{r},
\]

(1-4)

If we recall that the signal is really a sinusoid at a frequency of \( \omega_s \), the carrier frequency, we can write the electric field (E-field) at \( \Delta A \) as

\[
E = \frac{E_s}{r} e^{i\omega_s \tau},
\]

(1-5)

where \( \tau \) is the time the signal takes to propagate from the source to the area \( \Delta A \).
For the next step we invoke the relations \( r = r/c \), \( \omega_o = 2\pi f_o \), and \( f_o = c/\lambda \) where \( c \) is the speed of light and \( \lambda \) denotes wavelength. With this, we can write the electric field at \( \Delta A \) as

\[
E = \frac{E_o}{r} e^{j2\pi r/\lambda}.
\]  

(1-6)

Let us now turn our attention to determining the E-field at some \( \Delta A \) when we have the two radiators of Figure 1-2. We will use the geometry of Figure 1-3 as an aid in our derivation. We denote the upper radiator (point) of Figure 1-3 as radiator 1 and the lower radiator as radiator 2. The distances from the individual radiators to \( \Delta A \) are \( r_1 \) and \( r_2 \). We assume that the electric field intensity of each radiator is equal to \( \sqrt{P/2} \). Where \( P \) is the total power delivered to the radiators. The factor of 2 is included to denote the fact that the power is split evenly between the radiators (uniform weighting). In the above, we have made the tacit assumption that the radiation resistance is 1 ohm. The other terms that we will need are shown on Figure 1-3.

![Figure 1-3 – Geometry for two element radiator problem](image)

We can write the E-fields from the two radiators as

\[
E_1 = \frac{\sqrt{P/2}}{r_1} e^{j2\pi r_1/\lambda}
\]  

(1-7)

and

\[
E_2 = \frac{\sqrt{P/2}}{r_2} e^{j2\pi r_2/\lambda}.
\]  

(1-8)

From Figure 1-3, we can write \( r_1 \) and \( r_2 \) as

\[
r_1 = \sqrt{x_o^2 + (y_o - d/2)^2} = \sqrt{r^2 + d^2/4 - rd \sin \theta}
\]  

(1-9)

and
\[ r_1 = \sqrt{x_o^2 + (y_o + d/2)^2} = \sqrt{r^2 + d^2/4 + rd \sin \theta}. \]  

(1-10)

As indicated earlier, we will assume that \( r \gg d \). With this we can write

\[ r_1 \approx \sqrt{r^2 - rd \sin \theta} \approx r \left(1 - \frac{d}{2r} \sin \theta\right) \]  

(1-11)

and

\[ r_2 \approx \sqrt{r^2 + rd \sin \theta} \approx r \left(1 + \frac{d}{2r} \sin \theta\right). \]  

(1-12)

Where we have made use of the relation

\[ (1 \pm x)^N \approx 1 \pm Nx \quad \text{for} \quad x \ll 1. \]  

(1-13)

We note that since \( r_1 \) and \( r_2 \) are functions of \( \theta \), the E-fields will also be functions of \( \theta \). With this we write

\[ E_1(\theta) = \frac{\sqrt{P/2}}{r \left(1 - \frac{d}{2r} \sin \theta\right)} \exp\left(j2\pi \left(r - \frac{d}{2} \sin \theta\right)/\lambda\right) \]  

(1-14)

and

\[ E_2(\theta) = \frac{\sqrt{P/2}}{r \left(1 + \frac{d}{2r} \sin \theta\right)} \exp\left(j2\pi \left(r + \frac{d}{2} \sin \theta\right)/\lambda\right). \]  

(1-15)

In Equations 1-14 and 1-15, we can set the denominator terms to \( r \) since \( d/2r \ll 1 \). We can’t do this in the exponential terms because phase is measured modulo \( 2\pi \).

The total electric field at \( \Delta A \) is

\[ E(\theta) = E_1(\theta) + E_2(\theta) \]  

(1-16)

or

\[ E(\theta) = \frac{\sqrt{P/2}}{r} \exp\left(j2\pi \left(r - \frac{d}{2} \sin \theta\right)/\lambda\right) + \frac{\sqrt{P/2}}{r} \exp\left(j2\pi \left(r + \frac{d}{2} \sin \theta\right)/\lambda\right) \]

\[ = \frac{\sqrt{P/2}}{r} e^{j2\pi r/\lambda} \left(e^{-j\pi d \sin \theta}/\lambda + e^{j\pi d \sin \theta}/\lambda\right) \]

\[ = \frac{\sqrt{P/2}}{r} e^{j2\pi r/\lambda} \left(2 \cos\left(\frac{\pi d}{\lambda} \sin \theta\right)\right). \]  

(1-17)

At this point we define an antenna radiation pattern as (note: later we will define a directive gain pattern)

\[ R(\theta) = \frac{|E(\theta)|^2}{P/r^2}. \]  

(1-18)
With this, we obtain the radiation pattern for the dual, isotropic radiator antenna as

\[ R(\theta) = 2\cos^2\left(\frac{\pi d}{\lambda} \sin \theta\right). \]  

We will be interested in \( R(\theta) \) for \( |\theta| < \pi/2 \). We call the region \( |\theta| < \pi/2 \) visible space. Actually, physical visible space extends from \(-\pi\) to \(\pi\) but values of \( |\theta| > \pi/2 \) correspond to the back of an antenna, which is usually shielded.

Figure 1-4 contains plots of \( R(\theta) \) for three values of \( d \): \( d = \lambda \), \( \lambda/2 \) and \( \lambda/4 \). For \( d = \lambda \) there are three peaks in the radiation pattern: at \( 0 \), \( \pi/2 \), and \( -\pi/2 \). The peaks at \( \pm \pi/2 \) are termed grating lobes and are usually undesirable. For \( d = \lambda/4 \) the radiation pattern doesn’t return to zero and the width of the central region is broad. This is also a generally undesirable characteristic. The case of \( d = \lambda/2 \) is a good compromise that leads to a fairly narrow center peak and levels that go to zero at \( \pm \pi/2 \). In the design of phased array antennas we find that \( d = \lambda/2 \) is usually a desirable design criterion.

![Radiation Pattern for a two element array with various element spacings](image)

Figure 1-4 – Radiation Pattern for a two element array with various element spacings

The central region of the plots of Figure 1-4 is termed the main beam and the angle spacing between the 3-dB points (the points where the radiation pattern is down 3 dB from its peak value) is termed the beamwidth. It will be noted that the beamwidth is inversely proportional to the spacing between the radiators. This corresponds to our findings in the first radar course where we found that there was an inverse relation between beamwidth and the antenna diameter.

The problem we just solved is the transmit problem. That is, we supplied power to the radiators and determined how it was distributed on a sphere. We now want to consider the reverse problem and look at the receive antenna. This will illustrate an important problem called reciprocity. Reciprocity says that we can analyze an antenna either way and get the same radiation pattern. Stated another way, in general, the radiation pattern of an antenna is the same on transmit as on receive.
For this case we consider the two “radiators” of Figure 1-2 as receive antennas that are isotropic. Here we call them receive elements. We assume that an E-field radiates from a point that is located at a range $r$ from the center of the two receive elements. The receive elements are separated by a distance of $d$ as for the transmit case. The required geometry is shown in Figure 1-5.

The outputs of the receive elements are multiplied by $\sqrt{2}$ and summed. The voltage out of each element is proportional to the E-field at each element. It is represented as a complex number to account for the fact that the actual signal, which is a sinusoid, is characterized by an amplitude and phase.

![Figure 1-5 – Two element array, receive geometry](image)

The E-field at all points on a sphere (or a circle in two dimensions) has the same amplitude and phase. Also, since $d \ll r$ the sphere is a plane (a line in two dimensions – which is what we will use here) at the location of the receive elements. The line is oriented at an angle of $\theta$ relative to the vertical. We term this line the constant E-field line. $\theta$ is also the angle of the point from which the E-field radiates to the horizontal line of Figure 1-5. We term the horizontal line the antenna boresight. In more general terms, the antenna boresight is the normal to the plane containing the elements and is pointed generally toward the point from which the E-field radiates.

We define the vertical line through the elements as the reference line. From Figure 1-5, it is obvious that the spacing between these two lines and the reference line is $(d/2) \sin \theta$. If we define the E-field at the center point between the elements as

$$E = E_i e^{j2\pi \lambda r}$$

then the E-field at the elements will be

$$E_1(\theta) = E_i e^{j2\pi (r+(d/2) \sin \theta)/\lambda}$$

and

$$E_2(\theta) = E_i e^{j2\pi (r-(d/2) \sin \theta)/\lambda}.$$
Since the voltage out of each element is proportional to the E-field at each element, the voltages out of the elements are

\[ V_1(\theta) = V_i e^{i2\pi(r+(d/2)\sin\theta)/\lambda} \]  
\[ V_2(\theta) = V_i e^{i2\pi(r-(d/2)\sin\theta)/\lambda}. \]  

With this, the voltage at the summer output is

\[ V(\theta) = \frac{1}{\sqrt{2}} (V_1(\theta) + V_2(\theta)) = \frac{V_i}{\sqrt{2}} e^{i2\pi r/\lambda} 2\cos\left(\frac{\pi d}{\lambda} \sin \theta\right). \]

We define the radiation pattern as

\[ R(\theta) = \frac{|V(\theta)|^2}{V_r^2}, \]

which yields

\[ R(\theta) = 2\cos^2\left(\frac{\pi d}{\lambda} \sin \theta\right). \]

This is the same result that we got for the transmit case and serves to demonstrate that reciprocity applies to antennas. This will allow us to use either the receive or transmit approach when analyzing more complex antennas. We will use the technique that is easiest for the particular problem that we are addressing.

1.3 N-ELEMENT LINEAR ARRAY

We now want to extend the results of the previous section to a linear array of elements. A drawing of the linear array we will analyze is shown in Figure 1-6. As implied by the figure, we will use the receive approach to derive the radiation pattern for this antenna. The array consists of N elements with a spacing of \( d \) between the elements. The output of each element is weighted by a factor of \( a_k \) and the results summed to form the signal out of the antenna. In general, the weights, \( a_k \), can be complex (in fact, we will find that we can \textit{steer} the antenna beam by assigning appropriate phases to the \( a_k \)).
The slanted line originating at the first element depicts the plane wave that has arrived from a point target that is at an angle of $\theta$ relative to the boresight of the antenna. In our convention, $\theta$ is positive as shown in Figure 1-6. From Figure 1-6 it should be obvious that the distance between the plane wave and the $k^{th}$ element is

$$d_k = kd \sin \theta \quad 0 \leq k \leq N-1.$$  

(1-28)

This means that the E-field at the $k^{th}$ element is

$$E_k(\theta) = E_0 e^{j2\pi r/\lambda} e^{j2\pi d_k/\lambda} = E_0 e^{j2\pi r/\lambda} e^{j(2\pi kd \sin \theta)/\lambda}$$  

(1-29)

and that the voltage out of the $k^{th}$ element is

$$V_k(\theta) = a_k V_r e^{j2\pi r/\lambda} e^{j(2\pi kd \sin \theta)/\lambda}.$$  

(1-30)

With this, the voltage out of the summer is

$$V(\theta) = \sum_{k=0}^{N-1} V_k(\theta) = a_k V_r e^{j2\pi r/\lambda} e^{j(2\pi kd \sin \theta)/\lambda} = V_r e^{j2\pi r/\lambda} \sum_{k=0}^{N-1} a_k e^{j(2\pi kd \sin \theta)/\lambda}.\)  

(1-31)

For convenience, we will let $\Phi = 2\pi d \sin \theta/\lambda$ and write

$$V(\theta) = V_r e^{j2\pi r/\lambda} \sum_{k=0}^{N-1} a_k e^{jk\Phi}.$$  

(1-32)

As before, we define the radiation pattern as

$$R(\theta) = \frac{|V(\theta)|^2}{V_r^2}.$$  

(1-33)
which yields
\[
R(\theta) = \left| \sum_{k=0}^{N-1} a_k e^{jk\Phi} \right|^2.
\] (1-34)

We now want to consider the special case of a linear array with uniform weighting. For this case \( a_k = 1/\sqrt{N} \). If we consider the sum term we can write
\[
A(\theta) = \sum_{k=0}^{N-1} a_k e^{jk\Phi} = \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{jk\Phi} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{jk\Phi}.
\] (1-35)

At this point we invoke the relation
\[
\sum_{k=0}^{N-1} x^k = \frac{1 - x^N}{1 - x}
\] (1-36)
to write
\[
A(\theta) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{jk\Phi} = \frac{1}{\sqrt{N}} \frac{1 - e^{jN\Phi}}{1 - e^{j\Phi}}
\]
\[
= \frac{1}{\sqrt{N}} e^{jN\Phi/2} \left[ \frac{e^{jN\Phi/2} - e^{-jN\Phi/2}}{e^{j\Phi/2} - e^{-j\Phi/2}} \right],
\] (1-37)
\[
= \frac{1}{\sqrt{N}} e^{j(N-1)\Phi/2} \sin(N\Phi/2) / \sin(\Phi/2)
\]
Finally, we get
\[
R(\theta) = |A(\theta)|^2 = \frac{1}{N} \left( \frac{\sin(N\Phi/2)}{\sin(\Phi/2)} \right)^2 = \frac{1}{N} \left( \frac{\sin(N\frac{\pi}{\lambda} \sin \theta)}{\sin(\frac{\pi}{\lambda} \sin \theta)} \right)^2.
\] (1-38)

Figure 1-7 contains a plot of \( R(\theta) \) vs. \( \theta \) for \( N = 20 \) and \( d = \lambda, \lambda/2 \) and \( \lambda/4 \).

As with the two element case, it will be noted that grating lobes appear for the case where \( d = \lambda \). Also, it will be noted that the width of the mainlobe varies inversely with element spacing. As indicated earlier, this is expected because the larger element spacing implies a larger antenna which translates to a smaller beamwidth. It will be noted that the peak value of \( R(\theta) \) is 20, or \( N \), and occurs at \( \theta = 0 \). This value can also be derived by taking the limit of \( R(\theta) \) as \( \theta \to 0 \).
Figure 1-7 – Radiation pattern for an N-element linear array with different element spacings

For the general case where \(|a_i| \neq C\), where C is a constant, one must use

\[
R(\theta) = \left| \sum_{k=0}^{N-1} a_k e^{-j\phi} \right|^2, \quad \Phi = 2\pi d \sin \theta / \lambda \tag{1-39}
\]

to compute the radiation pattern. With modern computers, and software such as MATLAB®, this doesn’t pose much of a problem. It will be noted that the term inside of the absolute value is of the form of a discrete Fourier transform. This suggests that one could use the FFT to compute \(R(\theta)\). In fact, when we consider planar arrays we will discuss the use of the FFT to compute the antenna patterns.

1.4 ANTENNA GAIN PATTERN

The antenna radiation pattern is useful for determining such antenna properties as beamwidth, grating lobes and sidelobe levels. However, it does not provide an indication of antenna gain. To obtain this we want to define an antenna gain function. As its name implies, this function provides an indication of antenna gain as a function of angle. To be more specific, it provides an indication of the directive gain of the antenna as a function of angle. This is the gain we used in the radar-range equation and is also the gain we determine from Equations 1-1 and 1-2.

According to Chapter 6 of Skolnik’s Radar Handbook\(^1\) and Jasik’s antenna handbook\(^2\) we can write the antenna (directive) gain pattern\(^3\) as

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\(^3\) We distinguish between directive gain and directive gain pattern here. As we will discuss below, directive gain is a number. The directive gain pattern is a function of \(\theta\) and \(\phi\).

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\[ G(\theta, \phi) = \frac{\text{Radiation intensity on a sphere of radius } r \text{ at an angle } (\theta, \phi)}{\text{Average radiation intensity over a sphere of radius } r}, \quad (1-40) \]

or
\[ G(\theta, \phi) = \frac{R(\theta, \phi)}{4\pi r^2 \int R(\theta, \phi) \, d\Omega} = \frac{R(\theta, \phi)}{\bar{R}} \quad (1-41) \]

where \( d\Omega \) is a differential area on the sphere.

To compute the denominator integral we consider the geometry of Figure 1-8. In this figure, the vertical row of dots represents the linear array. From the figure, the differential area can be written as
\[ d\Omega = (du)ds = r^2 \cos \theta \, d\theta \, d\phi \quad (1-42) \]

and the integral in the denominator becomes
\[ \bar{R} = \frac{1}{4\pi r^2} \int_{\phi=0}^{\pi/2} \int_{\theta=-\pi/2}^{\pi/2} R(\theta, \phi) r^2 \cos \theta \, d\theta \, d\phi. \quad (1-43) \]

For the linear array we have \( R(\theta, \phi) = R(\theta) \) and
\[ \bar{R} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} R(\theta) \cos \theta \, d\theta. \quad (1-44) \]

For the special case of a linear array with uniform weighting we get
\[ \bar{R} = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left( \frac{\sin \left( 2\pi \frac{ld}{\lambda} \sin \theta \right)}{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)} \right)^2 \cos \theta \, d\theta \]
\[ = \left( \frac{\pi d}{\lambda} \sin \theta \right)^2 \int_{0}^{\pi/2} 1 \left( \frac{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)}{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)} \right)^2 \cos \theta \, d\theta \]
\[ = \frac{\pi}{2} \int_{0}^{\pi/2} 1 \left( \frac{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)}{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)} \right)^2 \cos \theta \, d\theta \]

where the last equality is a result of the fact that the integrand is an even function.

After considerable computation, it can be shown that
\[ \bar{R} = 1 + \frac{2}{N} \sum_{k=1}^{N-1} \sum_{l=1}^{k} \text{sinc} \left( 2ld/\lambda \right). \quad (1-46) \]
As a “sanity check” we want to consider a point-source (isotropic) radiator. This can be considered a special case of an $N$-element, uniformly-illuminated, linear array with an element spacing of $d = 0$. For this case we get $R(\theta) = 1$. Further, for $d = 0$,  
\[
\text{sinc}(2ld/\lambda) = 1 \quad \text{and}
\]
\[
\bar R = 1 + \frac{2}{N} \sum_{k=1}^{N-1} k = 1 + \frac{2}{N} \sum_{k=1}^{N-1} k = N.
\]  
(1-48)

This gives  
\[
G(\theta) = \frac{R(\theta)}{\bar R} = \frac{N}{\bar R} = 1.
\]  
(1-48)

It can also be shown that, for a general, $N$-element, uniformly-illuminated, linear array with an element spacing of $d = \lambda/2$, and weights of $a_k = 1/\sqrt{N}$, $\bar R = 1$ and  
\[
G(\theta) = R(\theta) = \frac{1}{N} \left( \frac{\sin \left( \frac{N\lambda d}{2} \sin \theta \right)}{\sin \left( \frac{\lambda d}{2} \sin \theta \right)} \right)^2.
\]  
(1-49)

For the case of a general, non-uniformly illuminated, linear array, $\bar R$ must be computed numerically from the Equation 1-44

We want to now consider the directive gain, $G$. We will define this as the maximum value of $G(\theta)$. For the uniformly illuminated, linear array considered above, $G = G(0)$. Figure 1-9 contains a plot of normalized $G$ vs. $d/\lambda$ for several values of $N$. In this plot, the normalized $G$ is $G/N$. 

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Figure 1-9 – Normalized Directive Gain vs. element spacing

The shapes of the curves in Figure 1-9 are very interesting, especially around integer multiples of $d/\lambda$. For example for $d/\lambda$ slightly less than 1, $G/N$ is between about 1.7 and 1.9, whereas, for $d/\lambda$ slightly greater than 1, $G/N$ is about 0.7. In other words, a very small change in element spacing causes the directive gain to vary by a factor of about 1.8/0.7 or 4 dB. The reason for this is shown in Figure 1-10 which contains a plot of $R(\theta)$ for $d/\lambda$ values of 0.9, 1.0 and 1.1 and a 20-element linear array with uniform weighting (uniform illumination). In this case $R(\theta)$ is plotted vs. $\sin \theta$ to better illustrate the widths of the grating lobes (the lobes not at 0).

For the case where $d/\lambda$ is 0.9 (blue curve), the radiation pattern does not contain grating lobes. This means that all of the transmitted power can be focused in the main beam. For the cases where $d/\lambda$ is either 1.0 or 1.1, the radiation pattern contains grating lobes. This means that some of the transmitted power will be taken away from the main lobe (the central region) to be sent to the grating lobes. This will thus reduce the directive gain of the antenna relative to the case where $d/\lambda$ is 0.9. Furthermore, since there are two grating lobes for $d/\lambda = 1.1$ and only one grating lobe for $d/\lambda = 1.0$ ($\frac{1}{2}$ lobe at $\sin \theta = 1$ and $\frac{1}{2}$ lobe at $\sin \theta = -1$), the gain will be less when $d/\lambda$ is 1.1 than when it is 1.0.

From Figure 1-9, we would expect similar behavior of the directive gain for values of $d/\lambda$ near other integer values. However, the variation in gain as $d/\lambda$ transitions from below to above an integer values decreases as the integer value of $d/\lambda$ increases.
1.5 BEAMWIDTH, SIDELOBES AND AMPLITUDE WEIGHTING

*Figure 1-11* contains a plot of $G_d(\theta)$ for a 20 element array with an element spacing of $d/\lambda = 0.5$ and uniform weighting. In this case, the units on the vertical scale are in dBi. The unit notation dBi stands for *dB relative to an isotropic radiator*, and says that the directive gain is referenced to the gain of an isotropic radiator, which is unity.

As discussed earlier, the lobe near $\theta = 0$ is termed the *main beam*. The lobes surrounding the main beam are the *sidelobes*. The first couple of sidelobes on either side of the main beam are termed the *near-in* sidelobes and the remaining sidelobes are...
termed the far-out sidelobes. For this antenna, the directive gain is 13 dB (10log(20)) and the near-in sidelobes are about 13 dB below the peak of the main beam (13 dB below the main beam). The far-out sidelobes are greater than 20 dB below the main beam.

The beamwidth is defined at the width of the main beam measured at the 3-dB points on the main beam. For the pattern of Figure 1-11 the beamwidth is 5 degrees.

The near-in sidelobe level of 13 dB is often considered undesirably high. To reduce this level, antenna designers usually apply an amplitude taper to the array by setting the $a_k$ to different values. Generally, the values of $a_k$ are varied symmetrically across the elements so that the elements on opposite sides of the center of the array have the same value of $a_k$. One usually tries to choose the $a_k$ so that one achieves a desired sidelobe level while minimizing the beamwidth increase and gain decrease usually engendered by weighting. The optimum weighting in this regard is Chebychev weighting. Up until recently, Chebychev weights were very difficult to generate. However, over the past 10 or so years, standard algorithms have become available. For example, the MATLAB® Signal Processing Toolbox has a built-in function (chebwin) for computing Chebychev weights. An approximation to Chebychev weighting is Taylor weighting. Taylor weights are a computed from a Taylor series approximation to Chebychev weights. An algorithm for computing Taylor weights is given in Appendix A.

In space-fed phased arrays and reflector antennas the amplitude taper is created by the feed and is applied on both transmit and receive. Since the amplitude taper is created by the feed, the type of taper is somewhat limited because of feed design limitations. In corporate or constrained feed phased arrays the taper is controlled by the way that power is delivered to, or combined from, the various elements. Again, this limits the type of amplitude taper that can be obtained. In solid state phase arrays, one has considerable flexibility in controlling the amplitude taper on receive. However, it is currently very difficult to obtain an amplitude taper on transmit because all of the transmit/receive (T/R) modules must be operated at full power for maximum efficiency. The aforementioned types of antennas and feed mechanisms will be discussed again later.

Figure 1-12 contains a plot of $G(\theta)$ for a 20-element linear array with $d/\lambda = 0.5$ and Chebychev weighting. The Chebychev weighting was chosen to provide a sidelobe level of 30 dB, relative to the main beam. As can be seen from Figure 1-12, the sidelobe level is, indeed, 30 dB below the peak main beam level. It will be noted that the directive gain at $\theta = 0$ (the directive gain or, simply, gain) is about 12.4 dB rather than the 13 dB gain associated with a 20-element, linear array with uniform illumination. Thus, the amplitude taper has reduced the antenna gain by about 0.6 dB. Also, the beamwidth of the antenna has increased to 6.2 degrees.
1.6 STEERING

Thus far, the antenna patterns we have generated all had their main beams located at 0 degrees. We now want to address the problem of placing the main beam at some desired angle. This is termed *beam steering*. We will first address the general problem of *time delay steering* and then develop the degenerate case of *phase steering*. The concept of beam steering, as discussed here, applies to phased array antennas. It does not apply to reflector antennas.

To address this problem, we refer to the N-element linear array geometry of *Figure 1-6*. Let the E-field from the point source be

\[ E_p(t) = \text{rect}\left(\frac{t}{\tau_p}\right)e^{j2\pi f_o t} \]  

(1-50)

where \( \tau_p \) is the pulse width, \( f_o \) is the carrier frequency and \( \text{rect}[x] \) is the rectangle function. We will further assume that the point source radiator is stationary and located at some range, \( R \).

The voltage out of the \( k^{th} \) antenna element (before the weighing, \( a_k \) ) is

\[ v_k(t) = \text{rect}\left(\frac{t - \tau_k}{\tau_p}\right)e^{j2\pi f_o (t - \tau_k)} \]  

(1-51)

where \( \tau_k \) is the time delay from the point source radiator to the \( k^{th} \) element and is given by

\[ \tau_k = \frac{R_k}{c} = \frac{R + kd \sin \theta}{c} = \tau_R + k\tau_{d\theta}. \]  

(1-52)
In the above, we have assumed that the voltage magnitude $V_r$ is unity (see page 9).

Instead of treating the weights, $a_k$, as multiplication factors we treat them as operators on the voltages at the output of the antenna elements. With this we write the voltage out of the summer as

$$V(\theta) = \sum_{k=0}^{N-1} a_k(v_k(t), k).$$

We want to determine how the weighting functions, $a_k(v_k(t), k)$, must be chosen so as to focus the beam at some angle $\theta_o$.  

*Figure 1-13* contains a sketch of the magnitudes of the various $v_k(t)$.  The main thing illustrated by this figure is that the pulses out of the various antenna elements are not aligned.  This means that the weighting functions, $a_k(v_k(t), k)$, must effect some desired alignment of the signals.  More specifically, the $a_k(v_k(t), k)$ must be chosen so that the signals out of the weighting functions are aligned (and in-phase) at some desired $\theta_o$.  To accomplish this, the $a_k(v_k(t), k)$ must introduce appropriate time delays (and possibly phase shifts) to the various $v_k(t)$. They must also appropriately scale the amplitudes of the various $v_k(t)$. This introduction of time delays to focus the beam at some angle $\theta_o$ is termed *time delay steering*.

![Figure 1-13 – Sketch of $|v_k(t)|$](image)

If we substitute for $\tau_k$ into the general $v_k(t)$ we get

$$v_k(t) = \text{rect} \left[ \frac{t - \tau_k}{\tau_p} \right] e^{j2\pi f_d(t - \tau_p)} = \text{rect} \left[ \frac{t - \tau_p - k\tau_d}{\tau_p} \right] e^{j2\pi f_d(t - \tau_p - k\tau_d)}. \quad (1-54)$$

To time align all of the pulses out of the weighting functions, the weighting function must introduce a time delay that cancels the $k\tau_d$ term in $v_k(t)$ at some $\theta_o$.  Specifically, $a_k(v_k(t), k)$ must be chosen such that

$$V_k(t) = a_k(v_k(t), k) = a_k|v_k(t + k\tau_d)| \quad (1-55)$$
where $\tau_{do} = \frac{d \sin \theta_o}{c}$. Using this with the $v_k(t)$ above results in

$$V_k(t) = |a_k| \text{rect} \left[ \frac{t - \tau_R - k(\tau_{do} - \tau_{do})}{\tau_p} \right] e^{i 2\pi f_o(t - \tau_k)}. \quad (1-56)$$

It will be noted that at $\theta = \theta_o$, $\tau_{do} = \tau_{do}$ and

$$V_k(t) = |a_k| \text{rect} \left[ \frac{t - \tau_R}{\tau_p} \right] e^{i 2\pi f_o(t - \tau_k)}. \quad (1-57)$$

In other words, the pulses out of the weighting functions are time aligned, and properly weighted.

Time delay steering is expensive and not easy to implement. It is needed in radars that use compressed pulse widths that are small relative to antenna dimensions. This can be seen from examining Figure 1-13. If $\tau_p$ is small relative to $(N - 1)\tau_{do}$ then, for some $\theta$, not all of the pulses will align. Stated another way, the pulse out of the first element will not be aligned with the pulse out of the $N^{th}$ element. However, this implies either a very small $\tau_p$ or a very large antenna (very large $(N - 1)\tau_{do}$). For example, if $\tau_p$ was 1 ns and the antenna was 2 m wide we would have $(N - 1)\tau_{do} = 6.7 \text{ ns} > \tau_p$ and time delay steering would be needed. However, if $\tau_p$ was 1 µs all of the pulses would be fairly well aligned and time delay steering would not be necessary.

Figure 1-14 contains a plot that gives an idea of the boundary between when time delay steering would and would not be necessary. The curve on this figure corresponds to the case where the antenna diameter, $D$, is 25% of the compressed pulse width. The choice of 25% is somewhat arbitrary but is probably representative of practical situations where the beam is steered to a maximum angle of 60 degrees.
The two regions of *Figure 1-14* indicate that the alternative to time delay steering is phase steering. Indeed, if we assume that the pulses are fairly well aligned then we can write

\[ V_k(t) = a(v_k(t), k) \]

\[ = |a_k| \text{rect} \left[ \frac{t - \tau_k}{\tau_p} \right] e^{j2\pi f_c t - \tau_k} e^{j2\pi f_c k \tau_d} v_k(t) \quad (1-58) \]

or that \( a_k = |a_k| e^{j2\pi f_c k \tau_d} \). This says that the weights, \( a_k \), modify the amplitudes and phases of the various \( v_k(t) \). This is why this technique is called *phase steering*.

Substituting for \( \tau_d \), in the phase term results in

\[ a_k = |a_k| e^{j2\pi k\tau_d} \sin \theta \lambda \cdot (1-59) \]

1.7 ELEMENT PATTERN

In the equations above it was assumed that all of the elements of the antenna were isotropic radiators. In practice antenna elements are not isotropic but have their own radiation pattern. This means that the voltage (amplitude and phase) out of each element depends upon \( \theta \), independent of the phase shift caused by the element spacing. If all of the elements are the same, and oriented the same relative to boresight, then the dependence voltage upon \( \theta \) will be the same for each element (again, ignoring the phase shift caused by the element spacing). In equation form, the voltage out of each element will be

\[ v_k(t) = A_{eb}(\theta) \text{rect} \left[ \frac{t - \tau_k}{\tau_p} \right] e^{j2\pi f_c t - \tau_k} \quad (1-60) \]

and the voltage out of the summer (assuming phase steering) will be

\[ V(\theta) = A_{eb}(\theta) e^{j2\pi f_c / \lambda} \sum_{k=0}^{N-1} a_k e^{jkd\sin \theta \lambda} = A_{eb}(\theta) e^{j2\pi f_c / \lambda} A_{array}(\theta) \quad (1-61) \]

The resulting radiation pattern will be

\[ R(\theta) = |V(\theta)|^2 = R_{eb}(\theta) R_{array}(\theta) \quad (1-62) \]

and the resulting directive gain pattern will be

\[ G(\theta) = G_{eb}(\theta) G_{array}(\theta) \quad (1-63) \]
In other words, to get the radiation or gain pattern of an antenna with a non-isotropic element pattern one multiplies the array pattern (found by the aforementioned techniques) by the radiation or gain pattern of the elements.

As a closing note, in general, the element pattern is not steered, only the array pattern.

1.8 PHASE SHIFTERS

In the above discussions, a tacit assumption is that the phase of each weight, \( a_k \), can take on a continuum of values. In practice, the phase can only be adjusted in discrete steps because the devices that implement the phase shift, the phase shifters, are digital. Typical phase shifters use 3 to 5 bits to set the phase shift. If \( B_\phi \) is the number of bits used in the phase shifter then the number of phases will be \( N_\phi = 2^{B_\phi} \). As an example, a 3-bit phase shifter will have 8 phases that range from 0 to \( (2 \pi/8) \). As we will see when we consider planar arrays, this phase quantization caused by the phase shifters can have a deleterious effect on the sidelobes when the beam is steered to other than boresight. Skolnik’s Radar Handbook has a discussion of phase shifters in Chapter 7.

1.9 COMPUTING ANTENNA PATTERNS USING THE FFT

Earlier we showed that we could write

\[
R(\theta) = |A(\theta)|^2
\]

where

\[
A(\theta) = \sum_{k=0}^{N-1} a_k e^{j2\pi kd\sin \theta / \lambda}.
\]

Equivalently, we could write

\[
R(\theta) = |B(\theta)|^2
\]

where

\[
B(\theta) = \sum_{k=0}^{N-1} a_k e^{-j2\pi kd\sin \theta / \lambda}.
\]

It will be noted that \( B(\theta) \) has the form of the Discrete-Time Fourier Transform (DTFT). Indeed, if we were to consider the \( a_k \) to be a discrete-time signal then we would write its DTFT as
\[ B(f) = \sum_{k=0}^{N-1} a_k e^{-j 2\pi k \Delta f} \quad (1-68) \]

where \( \Delta t \) is the time spacing between the \( a_k \) and \( f \) denotes frequency.

When we compare the above two equations we can make the following analogies
\[ \Delta t \leftrightarrow d \quad (1-69) \]
and
\[ f \leftrightarrow \sin \theta/\lambda. \quad (1-70) \]

We know that we can use the FFT to compute \( B(f) \). Thus, by analogy, we can also use the FFT to compute \( B(\theta) \). The trick is to properly interpret the FFT output.

For a time-frequency FFT, the frequency extent of the FFT output is
\[ \Delta F = 1/\Delta f \quad (1-71) \]
and the spacing between the FFT output taps is
\[ \Delta f = 1/(M \Delta t) \quad (1-72) \]
where \( M \) is the number of FFT taps, or the length of the FFT (usually a power of 2). For a response centered at 0 Hz, the frequencies associated with the \( M \) FFT taps are
\[ f \in \left[ -\frac{M}{2}, -\frac{M}{2} - 1 \right]\frac{1}{M \Delta t}. \quad (1-73) \]

By analogy, the total extent of the FFT output for the antenna case is
\[ \Delta (\sin \theta/\lambda) = \Delta u/\lambda = 1/d \quad (1-74) \]
where we have used \( u = \sin \theta \). The spacing between FFT output taps is
\[ \delta (\sin \theta/\lambda) = \delta u/\lambda = 1/(M \delta). \quad (1-75) \]
The \( u \)'s associated with the FFT taps are
\[ u = \sin \theta \in \left[ -\frac{M}{2}, -\frac{M}{2} - 1 \right]\frac{\lambda}{Md}. \quad (1-76) \]

As we increase \( M \) we obtain \( B(\theta) \)'s at finer angle scales. As with time-frequency FFT’s we increase \( M \) by zero-padding the input to the FFT.

In future discussions we let \( N_{el} \) be the number of elements in the phased array. We let \( M = 2^N \) be the number of points in the FFT. To zero-pad, we load the first \( N_{el} \) taps of the FFT with the \( a_k \) and set the remaining \( M - N_{el} \) taps to zero.
We want to examine the total extent of $\sin \theta$ at the FFT output. From the above, we note that
\[
|u| = |\sin \theta| \leq \frac{\lambda}{2d}.
\] (1-77)

If $d = \lambda/2$ we have that $|\sin \theta| \leq 1$ which means that the output taps cover all angles between $-\pi/2$ and $\pi/2$.

If $d < \lambda/2$ we note that $\lambda/2d > 1$ and thus that some values of $\sin \theta$ can have a magnitude greater than 1. This means that some of the FFT output taps do not correspond to real angles and that we need to ignore these taps when we plot the antenna pattern. Said another way, we keep the $M'$ FFT output taps that satisfy $|\sin \theta| \leq 1$.

If $d > \lambda/2$ then we have $\lambda/2d < 1$ and thus that the range of values of $\sin \theta$ does not extend to $\pm 1$. This means that the FFT does not generate the full antenna radiation pattern. To get around this problem we place "fake" elements between the real elements so as to reduce the effective $d$, $d_{\text{eff}}$, so that it satisfies $d_{\text{eff}} \leq \lambda/2$. We give the "fake" elements amplitudes of zero. If $d_{\text{eff}} < \lambda/2$ we must discard some of the FFT outputs as discussed above.

With the above we write the radiation pattern, in sine space, as
\[
R(u) = |B(u)|^2.
\] (1-78)

The antenna gain pattern, in sine space, is given by
\[
G(u) = R(u)/\bar{R}
\] (1-79)

where
\[
\bar{R} = \frac{\lambda}{2Md} \sum R(u)
\] (1-80)

and the sum is taken over the $M'$ FFT output taps that make-up the antenna pattern (see the above discussions). The form of $\bar{R}$ given here is an Euler approximation to Equation 1-44 with the substitution: $u = \sin \theta$. The factor of 2 in the denominator of the above equation comes from the factor of $1/2$ in Equation 1-44. To plot $R(\theta)$ or $G(\theta)$ one would make the substitution $\theta = \sin^{-1} u$.

1.9.1 Algorithm

Given the above, we can formulate the following algorithm

- If the antenna has an element spacing of $d > \lambda/2$, insert "fake" elements between the actual elements until the element spacing satisfies $d \leq \lambda/2$. Set the amplitudes of the "fake" elements to zero. The amplitudes of the "regular"
elements will be the \( a_k \) from above (1’s for uniform weighting, Hamming coefficients for Hamming weighting, phases for steering, etc.).

- Choose an FFT length, \( M \), that is 5 to 10 times larger than \( N_{el} \) and is a power of 2. If you had to add “fake” elements, the \( N_{el} \) used in this computation must include both the actual and “fake” elements.

- Take the FFT and perform the equivalent of the MATLAB FFTSHIFT function to place the zero tap in the center of the FFT output data array. Call the result \( B \).

- Compute \( u = \sin \theta = \left[ -M/2 : M/2 - 1 \right] \left( \lambda / Md \right) \) where \( d \) is the element spacing from the first step. The notation here and in the following is MATLAB notation.

- Compute \( k = \text{find}\left( \text{abs}(u) \leq 1 \right) \) (find is the MATLAB find function).

- Compute \( R = |B(k)|^2 \) and, if needed, \( G = R/(\text{sum}(R)*\lambda/(2Md)) \). (sum is the MATLAB sum function)

- Compute \( \theta = \text{asin}(u) \).

- Plot \( G \) or \( R \) vs. \( u \) or \( \theta \). If you plot \( G \) or \( R \) vs. \( \theta \) you will plotting the radiation or gain pattern in angle space. If you plot \( G \) or \( R \) vs. \( u \) you will plotting the radiation or gain pattern in sine space.
We now want to extend the results of linear arrays to planar arrays. In a planar array the antenna elements are located on some type of regular grid in a plane. An example that would apply to a rectangular grid is shown in Figure 1-15.

The array lies in the X-Y plane and the array normal, or boresight, is the Z-axis. The intersections of lines with the numbers by them are the locations of the various elements. The line located at the angles $\theta$ and $\phi$ points to the field point (the target on transmit or the source, which could also be the target, on receive). The field point is located at a range of $r$ that, as before, is very large relative to the dimensions of the array (far field assumption). In the coordinate system of Figure 1-15 the field point is located at

$$
(x_f, y_f) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi).
$$

(1-81)

The 00 element is located at the origin and the $mn^{th}$ element is located at $(md_x, nd_y)$ where $d_x$ is the spacing between elements in the $x$ direction and $d_y$ is the spacing between elements in the $y$ direction. With this and Equation 1-81 we can find the range from the $mn^{th}$ to the field point, $r_{mn}$ as

$$
r_{mn} = \sqrt{(x_f - md_x)^2 + (y_f - nd_y)^2} = r + md_x \sin \theta \cos \phi + nd_y \sin \theta \sin \phi
$$

(1-82)

where we have made use of the fact that $r$ is much greater than the dimensions of the antenna.
If we invoke reciprocity and consider the receive case (as we did for linear arrays) we can write the voltage out of the \( mn \)th element as

\[
V_{mn}(\theta, \phi) = V_r a_{mn} \exp \left( \frac{j2\pi r}{\lambda} \right) \exp \left[ -\frac{j2\pi}{\lambda} \left( m \frac{\lambda}{d_x} \sin \theta \cos \phi + n \frac{\lambda}{d_y} \sin \theta \sin \phi \right) \right] \tag{1-83}
\]

where \( a_{mn} \) is the weighting applied to the \( mn \)th element. Given that the outputs of all \( mn \) elements are summed to form the overall output, \( V(\theta, \phi) \), we get

\[
V(\theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} V_{mn}(\theta, \phi) = V_r \exp \left( \frac{j2\pi r}{\lambda} \right) \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} \exp \left[ -\frac{j2\pi}{\lambda} \left( m \frac{\lambda}{d_x} \sin \theta \cos \phi + n \frac{\lambda}{d_y} \sin \theta \sin \phi \right) \right] \tag{1-84}
\]

where \( M+1 \) is the number of elements in the \( x \) direction and \( N+1 \) is the number of elements in the \( y \) direction. If we divide by \( V_r \) and ignore the phase (see the sections on linear arrays) we can write

\[
A(\theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} \exp \left[ -\frac{j2\pi}{\lambda} \left( m \frac{\lambda}{d_x} \sin \theta \cos \phi + n \frac{\lambda}{d_y} \sin \theta \sin \phi \right) \right]. \tag{1-85}
\]

At this point we adopt a notation that is common in phased array antennas, and consistent with the notation we used for linear arrays: sine space. We define

\[
u = \sin \alpha = \sin \theta \cos \phi \tag{1-86}
\]

and

\[
v = \sin \beta = \sin \theta \sin \phi. \tag{1-87}
\]

With this we write

\[
A(\sin \alpha, \sin \beta) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} \exp \left[ -\frac{j2\pi}{\lambda} \left( m \frac{\lambda}{d_x} \sin \alpha + n \frac{\lambda}{d_y} \sin \beta \right) \right] \tag{1-88}
\]

or

\[
A(u, v) = \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} \exp \left[ -\frac{j2\pi}{\lambda} \left( m \frac{\lambda}{d_x} u + n \frac{\lambda}{d_y} v \right) \right]. \tag{1-89}
\]

Consistent with our work on linear arrays we write the radiation pattern as

\[
R(u, v) = |A(u, v)|^2 \tag{1-90}
\]

and the directive gain as

\[
G(u, v) = R(u, v)/R \tag{1-91}
\]

where \( R \) will be discussed later.
When we plot \( G(u,v) \) we say we are plotting the directive gain pattern in sine space. When we plot \( G(\theta,\phi) \) we say that we are plotting the directive gain pattern in angle space. The usual procedure for finding \( G(\theta,\phi) \) is to first find \( G(u,v) \) and then perform a transformation from \( u,v \) space to \( \theta,\phi \). To derive the transformation, we can solve the \( u,v \) equations above for \( \theta \) and \( \phi \). Doing so results in

\[
\theta = \sin^{-1}\left(\sqrt{u^2 + v^2}\right) \tag{1-92}
\]

and

\[
\phi = \tan^{-1}\left(v/u\right) \tag{1-93}
\]

where the arctangent is the four-quadrant arctangent. With the above \( 0 \leq |\theta| \leq \pi \) and \(-\pi \leq \phi \leq \pi \), which covers the entire sphere.

An obvious constraint from the equation for \( \theta \), and the definition of \( u \) and \( v \), is \(|u| \leq 1, |v| \leq 1 \) and \( u^2 + v^2 \leq 1 \). This is worth noting because, as we found for the linear array, we will compute values of \( A(u,v) \) for \( u,v \) values that violate this constraint. As before, our solution will be to ignore \( A(u,v) \) for \( u,v \) values that violate the above constraint.

1.10.1 Elevation and Azimuth Cuts

As a final note, if one wants to plot an elevation cut of the directive gain pattern one would plot \( G(\theta,\pi/2) = G(u,v)|_{u=0} \). If one wanted an azimuth cut one would plot \( G(\theta,0) = G(u,v)|_{v=\sin^{-1}\theta} \). This derives from the fact that an elevation cut is a plot of the directive gain in the Y-Z plane of Figure 1-15 and an azimuth cut is a plot of the directive gain in the X-Z plane of Figure 1-15.

1.10.2 Weights for Beam Steering

In the equation for \( A(u,v) \), the \( a_{mn} \) are the weights that are used to provide a proper taper and to steer the beam. They are of the general form

\[
a_{mn} = |a_{mn}| \exp\left[\frac{j2\pi}{\lambda} (md_xu_o + nd_yv_o) \right] \tag{1-94}
\]

where \( u_o,v_o \) are the desired beam angles in sine space.
1.10.3 Use of the FFT to Compute Planar Array Patterns

As with the linear array, we recognize that $A(u,v)$ has the form of a Fourier transform, albeit a two-dimensional Fourier transform. Analogous to the technique for linear arrays, we can use the 2-D FFT to compute $A(u,v)$. The basic technique is the same. Namely:

- Put the antenna on a rectangular grid with spacings of $d_x \leq \lambda/2$ and $d_y \leq \lambda/2$. This could require adding dummy elements as you did for the linear array. We will discuss this further shortly. This essentially requires specifying all of the $a_{mn}$, even the dummy elements (where $|a_{mn}| = 0$). As indicated above, this is also where the beam steering is performed.
- Take the 2-D FFT. As before, the FFT should be a power of 2 and should be 5 to 10 times larger, in each direction (X and Y), than the array. The lengths in the X and Y direction do not need to be the same.
- Set $A(u,v)$ to zero for $|u| > 1$, $|v| > 1$ or $u^2 + v^2 > 1$
- Find $R(u,v)$ and $G(u,v)$ and make the appropriate plots.

1.10.4 Array Shapes and Element Locations (Element Packing)

The work above was developed for the case of a rectangular array with the elements placed on a rectangular grid. This is not a common method of constructing antenna. Many antennas are non-rectangular (circular or elliptical) and their elements are not placed on a rectangular grid (rectangular packing). In both cases the deviations from rectangular shape and/or rectangular packing are usually made to conserve array elements and increase the efficiency of the antenna (the elements at the corners or rectangular arrays do not contribute much to the antenna gain and cause the ridges you will note in your homework. You will look at the effects of non-rectangular arrays as part of the homework problems.

The most common element packing scheme besides rectangular packing is called triangular packing. The origin of the phrases will become clear in the ensuing discussions. Figure 1-16 contains sketches of sections of a planar array with rectangular and triangular packing. The dashed elements in the triangular packing illustration are dummy elements that must be included when one uses the 2-D FFT to compute the radiation pattern for an array with triangular packing. The need for the dummy elements stems from the fact that the FFT method must use rectangular packing. The amplitudes of the dummy elements are set to zero (as was done when we wanted to analyze arrays with elements spacings that were greater than $\lambda/2$).

In the triangular packing, the elements are arranged in a triangular pattern. Thus the origin of the packing nomenclature.
1.10.5 Amplitude Weighting

As with linear arrays we can use amplitude weighting to reduce sidelobes. We use the same types of amplitude weightings as for linear arrays (Taylor, Chebychev, Hamming, Gaussian, etc.). The difference is that we now need to be concerned with applying the weightings in two dimensions. There are two basic ways to do this:

1. multiplicative weighting and
2. circularly symmetric weighting.

For the multiplicative method we would write the magnitudes of the weights as

\[ |a_{mn}| = |a_{m}|a_{n}. \]  \hspace{1cm} (1-95)

This method of determining the weights is the easiest of the two discussed here. It will provide predictable sidelobe levels on the principal planes (u cut and v cut) but not in the sidelobe regions between the principal planes.

To achieve predictable sidelobe levels over the entire sidelobe region one must use circularly symmetric weighting. To do this one can use the following procedure, which works well for circular arrays and reasonably well for elliptic arrays.

1. Generate a set of appropriate weights that have a number of elements that is equal to \( N_{wr} = L_{max}/d_{min} \) where \( L_{max} \) is the maximum antenna dimension and \( d_{min} \) is the minimum element spacing. Define an array of numbers, \( x_w \) that goes from -1 to 1 and has \( N_{wt} \) elements.

2. Find the location of all of the antenna elements relative to the center of the array. Let \( d_{xmn} \) and \( d_{ymn} \) be the x and y locations of the \( mn^{th} \) element relative to the center of the array. Let \( 2D_x \) and \( 2D_y \) be the antenna widths in the x and y

---

Figure 1-16 – Illustration of rectangular and triangular element packing
directions. Find the normalized distance from the center of the array to the $mn$th element using

$$x_{mn} = \sqrt{\left(\frac{d_{xmn}}{D_x}\right)^2 + \left(\frac{d_{ymn}}{D_y}\right)^2}.$$  

(1-96)

3. Use $x_{mn}$ to interpolate into the array of weights vs. $x_n$ to get the $|a_{mn}|$.

1.11 FEEDS

An antenna feed is the mechanism by which the energy from the transmitter is conveyed to the array so that it can be radiated into space. On receive, it is used to collect the energy from the array elements. There are two broad classes of feed types used in phased arrays: space feed and corporate, or constrained, feed. These two types of feed mechanisms are illustrated in Figures 1-17 and 1-18.

In a space feed the feed is some type of small antenna that radiates the energy to the array, through space. The feed could be a horn antenna or even another, smaller phased array. In a space fed array, the feed generates an antenna pattern, on transmit, which is captured by small antennas on the feed side of the array. These are represented by the v-shaped symbols on the left side of the array of Figure 1-17. The outputs of the small antennas undergo a phase shift (represented by the circles with $\phi$ in them) and are radiated into space by the small antennas represented by the v-shaped symbol on the right of the array. On receive, the reverse of the above occurs:

- The antennas on the right of the array capture the energy from the source
- The phase shifters apply appropriate phase shifts
- The antennas on the left of the array focus and radiate the energy to the feed
- The feed sends the energy to the receiver.
The phase shifters provide the beam steering as indicated in previous discussions. They also perform what is called a *spherical correction*. The E-field radiated from the feed has constant phase on a sphere, which is represented by the arcs in Figure 1-17. This means that the phase at each of the phase shifters will be different. This must be accounted for in the setting of the phase shifters. This process of adjusting the phase to account for the spherical wave front is termed spherical correction.

Generally, the feed produces its own gain pattern. This means that the signals entering each of the phase shifters will be at different amplitudes. This means that the feed is applying the amplitude weighting, $|a_{mn}|$, to the array. Generally, the feed gain pattern is adjusted to achieve a desired sidelobe level for the overall antenna. Feed patterns are typically shaped like part of one lobe of a sine/cosine function. In order to obtain a good tradeoff between gain and sidelobe levels for the overall antenna, the feed pattern is such that the level at the edge of the array is between 10 and 20 dB below the peak value. This is usually termed an edge taper. A feed that provides a 20 dB edge taper will result in lower array sidelobes than a feed that provides a 10 dB edge taper. However, a space-fed phased array with a 10 dB edge taper feed will have higher gain than a space-fed phased array with a 20 dB edge taper feed.
In a corporate feed phased array the energy is routed from the transmitter, and to the receiver, by a waveguide network. This is represented by the network of connections to the left of the array in Figure 1-18. In some applications the waveguide network can be structured to provide an amplitude taper to reduce sidelobes.

The phase shifters in a corporate feed array must include additional phase shifts to account for the different path lengths of the various legs of the waveguide network.

Generally, space feed phased arrays are less expensive to build because they don’t require the waveguide network that is required by the corporate feed phased array. However, the corporate feed phased array is smaller than the space feed phased array. The space feed phased array is generally as deep as it is tall or wide to allow for proper positioning of the feed. The depth of a corporate feed phased array is only about twice the depth of the array portion of a space feed phased array. The extra depth is needed to accommodate the waveguide network. Finally, the corporate feed phased array is more rugged than the space fed array since almost all hardware is on the array structure.

A sort of “limiting” case of the corporate feed phased array is the solid state phased array. For this array, the phase shifters of Figure 1-18 are replaced by solid state transmit/received (T/R) modules. The waveguide network can be replaced by cables since they carry only low power signals. The transmitters in each of the T/R modules are fairly low power (10 to 100 watt). However, a solid state phased array can contain thousands (3000 to 12,000) of T/R modules so that the total transmit power is comparable to that of at space feed phased array.
1.12 POLARIZATION

Thus far in our discussions we have played down the role of the electric field (E-field) in antennas. As our final topic we need to discuss E-fields for the specific purpose of discussing polarization. As you may have learned in your electromagnetic waves courses, E-fields have both direction and magnitude (and frequency). In fact, an E-field is a vector that is a function of both spatial position and time. If we consider a vector E-field that is traveling in the $z$ direction of a rectangular coordinate system we can express it as

$$\vec{E}(t, z) = E_x(t, z)\hat{a}_x + E_y(t, z)\hat{a}_y$$

(1-97)

where $\hat{a}_x$ and $\hat{a}_y$ are unit vectors. This formulation makes the assumption that the electric field is normal to the direction of propagation, $z$ in this case. In fact this is not necessary and we could have been more general by including a component in the $\hat{a}_z$ direction. A graphic showing the above E-field is contained in Figure 1-19. In this drawing, the $z$ axis is the LOS (line of sight) vector from the radar to the target. The $x$-$y$ plane is in the neighborhood of the face of the antenna. The $y$ axis is generally up and the $x$ axis is oriented so as to form a right-handed coordinate system. This is the configuration for propagation from the antenna to the target. When considering propagation from the target, the $z$ axis points along the LOS from the target to the antenna, the $y$ axis is up and the $x$ is again oriented so as to form a right-handed coordinate system.

When we speak of polarization we are interested in how the E-field vector, $\vec{E}(t, z)$, behaves as a function of time for a fixed $z$, or as a function of $z$ for a fixed $t$. To proceed further we need to write the forms of $E_x(t, z)$ and $E_y(t, z)$. We will use the simplified form of sinusoidal signal. With this we get

$$\vec{E}(t, z) = E_{xo}\sin[2\pi(f_o t + z/\lambda)]\hat{a}_x + E_{yo}\sin[2\pi(f_o t + z/\lambda) + \phi]\hat{a}_y.$$  

(1-98)

In the above $E_{xo}$ and $E_{yo}$ are positive numbers and represent the electric field strength. $f_o$ is the carrier frequency and $\lambda$ is the wavelength, which is related to $f_o$ by $\lambda = c/f_o$. $\phi$ is a phase shift that is used to control polarization orientation.
If $\mathbf{E}(t, z)$ remains fixed in orientation as a function of $t$ and $z$ the E-field is said to be linearly polarized. In particular

- If $\phi = 0$, $E_{xo} \neq 0$ and $E_{yo} = 0$ we say that the E-field is horizontally polarized.
- If $\phi = 0$, $E_{yo} \neq 0$ and $E_{xo} = 0$ we say that the E-field is vertically polarized.
- If $\phi = 0$, and $E_{xo} = E_{yo} \neq 0$ we say that the E-field has a slant 45° polarization.
- If $\phi = 0$, and $E_{xo} \neq E_{yo} \neq 0$ we say that the E-field has a slant polarization at some angle other than 45°. The polarization angle is given by $\tan^{-1}\left(E_{yo}/E_{xo}\right)$.

- If $\phi = \pm \pi/2$ and $E_{xo} = E_{yo} \neq 0$ we say that we have circular polarization. If $\phi = +\pi/2$ the polarization is left-circular because $\mathbf{E}(t, z)$ rotates counterclockwise, or to the left, as $t$ or $z$ increase. If $\phi = -\pi/2$ the polarization is right-circular because $\mathbf{E}(t, z)$ rotates clockwise, or to the right, as $t$ or $z$ increase.
- If $\phi$ is any other angle besides $\pm \pi/2$, $0$ or $\pi$ and/or $E_{xo} \neq E_{yo} \neq 0$ we say that the polarization is elliptical. It can be left ($\phi = +\pi/2$) or right ($\phi = -\pi/2$) elliptical.

As a note, polarization is always measured in the direction of propagation of the E-field to/from the antenna from/to the target. This is usually also the boresight angle. However, if one is looking at a target through the antenna sidelobes the direction of propagation is not the boresight. When polarization of an antenna is specified, it is the polarization in the main beam. The polarization in the sidelobes can be dramatically different than the polarization in the main beam.
1.13 REFLECTOR ANTENNAS

Older radars, and some modern radars where cost is an issue, use reflector types of antennas rather than phased arrays. Reflector antennas are much less expensive than phased arrays (thousands to hundreds of thousands of dollars as opposed to millions or tens of millions of dollars). They are also more rugged than phased arrays and are generally easier to maintain. They can be designed to achieve very good gain and very low sidelobes. The main disadvantages of reflector antennas, compared to phased array antennas, are that they must be mechanically scanned. This means that radars that employ reflector antennas will have limited multiple target capability. In fact, most target tracking radars that employ reflector antennas can track only one target at a time. Search radars that employ reflector antennas can detect and track multiple targets but the track update rate is limited by the scan time of the radar, which is usually on the order of 5 to 20 seconds. This, in turn, limits the track accuracy of these radars.

Another limitation of radars that employ reflector antennas is that separate radars are needed for each function. Thus, separate radars would be needed for search, track, and missile guidance. This requirement for multiple radars leads to interesting tradeoffs in radar system design. With a phased array it may be possible to use a single radar to perform the three aforementioned functions. Thus, while the cost of a phased array is high, relative to a reflector antenna, the cost of three radars with reflector antennas may be even more expensive than a single phased array radar. When one also accounts for factors such as cost of operators, maintenance, and other logistic issues, the cost tradeoff becomes even more interesting.

Almost all reflector antennas use some variation of a paraboloid. An example of such an antenna is shown in Figure 1-20. The feed shown in Figure 1-20 is located at the focus of the parabolic reflector. Since it is in the front, this antenna would be termed a front fed antenna. The lines from the reflector to the feed are struts that are used to keep the feed in place.

Skolnik’s radar handbook and Jasik’s antenna handbook have drawings of several variants on the paraboloid type antenna. In almost all of these, the reflector is formed by cutting off the top and/or bottom of the reflector, and sometimes the sides. Thus, the reflectors are portions of a paraboloid.

A parabola is used as a reflector because of its focusing properties. This is somewhat illustrated by Figure 1-21. In this figure it will be noted that the feed is at the focus of the parabola. From, analytic geometry we know that if rays emanate from the focus and are reflected off of the parabola, the reflected rays will be parallel. In this way parabolic antenna focuses the divergent E-field from the feed into a concentrated E-field. Stated another way, the parabolic reflector collimates the feed’s E-field.
As with space fed phased arrays, the feed pattern can be used to control the sidelobe levels of a reflector antenna. It does this by concentrating the energy at center of the reflector and causing it to taper off toward the edge of the reflector.

The process of computing the radiation pattern for a reflector antenna, where the feed is at the focus is fairly straight forward. Referring to Figure 1-21, one places a hypothetical plane parallel to the face of the reflector, usually at the location of the feed. This plane is termed the aperture plane. One then puts a grid of points in this plane. The points are typically on rectangular grid and are spaced $\lambda/2$ apart. The boundary of the points will be in a circle that follows the edge of the reflector. These points will be used as elements in a hypothetical phased array.

We think of the points, pseudo array elements, as being in the $x$-$y$ plane whose origin is at the feed. The $z$ axis of this coordinate system is normal to the aperture plane.

If we draw a line, in the $x$-$y$ plane, from the origin to the point $(x, y)$ the angle it makes with the $x$ axis is

$$
\phi = \tan^{-1}\left(\frac{y}{x}\right)
$$

(1-99)

where the arctangent is the four-quadrant arctangent. The distance from the origin to the point will be

$$
r = \sqrt{x^2 + y^2}.
$$

(1-100)

Now, one can draw a line from the point, perpendicular to the aperture plane, to the reflector. Examples of this are the lines $d_1l_1$ and $d_2l_2$ in Figure 1-21. The next step is to find the angle, $\theta$, between the $z$ axis and the point on the reflector.
From Figure 1-21 it will be noted that

\[ d + l = 2f \]  \hspace{1cm} (1-101)

where \( f \) is the focus of the parabola. Also,

\[ r^2 + l^2 = d^2. \]  \hspace{1cm} (1-102)

With this we can solve for \( d \) to yield

\[ d = \frac{1}{4f} \sqrt{r^2 + f^2} \]  \hspace{1cm} (1-103)

and

\[ \theta = \sin^{-1} \left( \frac{r}{d} \right). \]  \hspace{1cm} (1-104)

The angles \( \phi \) and \( \theta \) are next used to find the gain of the feed at the point where the ray intersects the parabolic reflector. This gain gives the amplitude of the pseudo element at \((x, y)\).

The above process is repeated for all of the pseudo elements in the aperture plane. Finally, the reflector antenna radiation pattern is found by treating the pseudo elements in the aperture plane as a planar phased array.

There is no need to be concerned about the phase of each element since the distance from the feed to the reflector to all points in the aperture plane are the same. This means that the various rays from the feed take the same time to get to the aperture plane. This further implies that the E-fields along each array will have the same phase in the aperture plane.
If the feed is not located at the focus of the parabolid the calculations needed to find the amplitudes and phases of the E-field at the pseudo elements become considerably more complicated. It is well beyond the scope of this class.
APPENDIX A – AN EQUATION FOR TAYLOR WEIGHTS

The following is an equation for calculating Taylor weights for an array antenna. It is similar to the equation on page 20-8 of “Antenna Engineering Handbook – Third Edition” by Richard C. Johnson, with some clarifications and corrections.

The un-normalized weights for the \( k^{th} \) element of \( K \) element linear array is

\[
a_k = 1 + 2 \sum_{n=1}^{\bar{n}-1} F(n,A,\bar{n}) \cos \left( \frac{2n\pi x_k}{K} \right) \quad (A-1)
\]

where

\[
F(n,A,\bar{n}) = \frac{\left[ (\bar{n} - 1)! \right]^2 \prod_{m=1}^{\bar{n}-1} \left( 1 - \frac{n^2}{\sigma^2 \left[ A^2 + (m - \frac{1}{2})^2 \right]} \right)}{(\bar{n} - 1 + n)! (\bar{n} - 1 - n)!}, \quad (A-2)
\]

\[
A = \frac{\cosh^{-1}(R)}{\pi}, \quad (A-3)
\]

\[
\sigma^2 = \frac{\bar{n}^2}{A^2 + (\bar{n} - \frac{1}{2})^2}, \quad (A-4)
\]

and

\[
R = 10^{SL/20}. \quad (A-5)
\]

\( SL \) is the desired sidelobe level, in dB, relative to the peak of the main beam. It is a positive number. For example, for a sidelobe level of -30 dB, \( SL = 30 \). This says that the sidelobe is 30 dB below the peak of the main beam.

\( \bar{n} \) is the number of sidelobes on each side of the main beam that one desires to have a level of approximately \( SL \) below the main beam peak amplitude.

The \( x_k \) can be computed using the following MATLAB notation

\[
z = [-K : K]/2;
\]

\[
x = z(2 : 2 : end);
\]

Finally, one needs to normalize the weights by dividing all of the \( a_k \) by \( \max_k(a_k) \).
APPENDIX B – CALCULATION OF $\vec{R}$ FOR A PLANAR ARRAY

In Section 1.4 we derived an equation for $\vec{R}$ for a linear array. We further extended this equation in Section 1.7 to allow us a method of calculating $\vec{R}$ when we had the radiation pattern of a linear array expressed in sine space. In this appendix, we want to derive an equation that will allow us to compute $\vec{R}$ for a planar array when we have the radiation for a planar array expressed in sine space.

The geometry we need to solve this problem is shown in Figure B-1. This figure is similar to Figure 1-8 where we have replaced the linear array (the column of dots) by a planar array (several columns of dots). We have also redefined the angles. We had to do this so as to not confuse the angles used to derive $\vec{R}$ with those used in deriving the equation for $R(u,v)$. When we derived Equation 1-43 we did not make this distinction. As a result, we used the same angle definitions ($\theta$ and $\phi$) to derive the equation for $\vec{R}$ as we did when we derived the equation for $R(u,v)$. We should not have done this since the angles are defined differently. We will re-derive Equation 1-42 and Equation 1-43 using the angles ($\alpha$ and $\varepsilon$) defined in Figure B-1. I chose the angles $\alpha$ and $\varepsilon$ because, if the array is vertical as shown, $\alpha$ is azimuth and $\varepsilon$ is elevation.

We start with the definition

$$\vec{R} = \frac{1}{4\pi r^2} \int_{\text{sphere}} R(\alpha, \varepsilon) d\Omega.$$  \hfill (B-1)

From Figure B-1 we write

$$d\Omega = (dw) ds = r^2 \cos \varepsilon \, d\alpha d\varepsilon.$$  \hfill (B-2)
Substituting Equation B-2 into Equation B-1 yields

\[
\bar{R} = \frac{1}{4\pi r^2} \int_{\epsilon=\pi/2}^{\pi/2} \int_{\alpha=0}^{2\pi} R(\alpha, \epsilon) r^2 \cos \epsilon \, d\alpha d\epsilon . \tag{B-3}
\]

At this point we recognize that a practical planar array only radiates in its forward hemisphere. This means that \( R(\alpha, \epsilon) \) will be zero for \( \alpha \in [\pi, 2\pi) \). If we use this, and cancel the \( r^2 \) terms, we get

\[
\bar{R} = \frac{1}{4\pi} \int_{\epsilon=\pi/2}^{\pi/2} \int_{\alpha=0}^{\pi} R(\alpha, \epsilon) \cos \epsilon \, d\alpha d\epsilon . \tag{B-3}
\]

For the next step we need to relate \( \alpha \) and \( \epsilon \) to \( u \) and \( v \). To do so, we want to write the triple \( (x_f, y_f, z_f) \) of Equation 1-81 in terms of the angles \( \alpha \) and \( \epsilon \). In Figure B-1 the center of the patch is the point \( (x_f, y_f, z_f) \). Thus, we can write

\[
(x_f, y_f, z_f) = (r \sin \epsilon, r \cos \epsilon \cos \alpha, r \cos \epsilon \sin \alpha) . \tag{B-4}
\]

By analogy to Equations 1-86 through 1-90 we can find that

\[ u = \sin \epsilon \tag{B-5} \]

and

\[ v = \cos \epsilon \cos \alpha . \tag{B-6} \]

and that

\[ R(u, v) = R(\sin \epsilon, \cos \epsilon \cos \alpha) . \tag{B-7} \]

Using this we can write Equation B-3 as

\[
\bar{R} = \frac{1}{4\pi} \int_{\epsilon=\pi/2}^{\pi/2} \int_{\alpha=0}^{\pi} R(\sin \epsilon, \cos \epsilon \cos \alpha) \cos \epsilon \, d\alpha d\epsilon . \tag{B-8}
\]

We now make the substitution \( u = \sin \epsilon \). This leads to \( du = \cos \epsilon \, d\epsilon \) and

\[
\bar{R} = \frac{1}{4\pi} \int_{u=-1}^{1} \int_{\alpha=0}^{\pi} R\left(u, \sqrt{1-u^2} \cos \alpha\right) \, d\alpha du , \tag{B-9}
\]

where we have made use of the fact that we can write \( \cos \epsilon = +\sqrt{1-\sin^2 \epsilon} \) since \( \epsilon \in [-\pi/2, \pi/2] \).

We next make the change of variables \( v = \sqrt{1-u^2} \cos \alpha \). This leads to \( dv = -\sqrt{1-u^2} \sin \alpha \, d\alpha \). We write the term multiplying \( d\alpha \) as \( \cos \epsilon \sin \alpha \) and use Equation B-4 to note that

\[ \sin^2 \epsilon + \cos^2 \epsilon \cos^2 \alpha + \cos^2 \epsilon \sin^2 \alpha = 1 \tag{B-10} \]
which leads to
\[
\cos \varepsilon \sin \alpha = \sqrt{1 - \sin^2 \varepsilon} \cos \alpha \cos \alpha = \sqrt{1 - u^2 - v^2}.
\] (B-11)
and
\[
d\alpha = \frac{-dv}{\sqrt{1 - u^2 - v^2}}.
\] (B-12)

Next we note that when \( \alpha = 0, \ v = \sqrt{1 - u^2} \) and when \( \alpha = \pi, \ v = -\sqrt{1 - u^2} \). If we substitute all of this into Equation B-9 we get
\[
\bar{R} = \frac{1}{4\pi} \int_{u=-1}^{1} \int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} R(u,v) \frac{-dv}{\sqrt{1 - u^2 - v^2}} du
\] (B-13)
or
\[
\bar{R} = \frac{1}{4\pi} \int_{u=-1}^{1} \int_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{R(u,v)}{\sqrt{1 - u^2 - v^2}} dv du.
\] (B-14)

We recognize the above as the integral of \( R(u,v)/\sqrt{1 - u^2 - v^2} \) over a circle of unit radius in the \((u,v)\) plane. When we compute \( R(u,v) \) using the FFT, or by direct computation, we are computing it only on a unit radius circle in the \((u,v)\) plane. (Recall that when we computed \( R(u,v) \) using the FFT, we zeroed the values of \( R(u,v) \) that were not in a unit circle.) This means that we can compute a good approximation to \( \bar{R} \) by dividing each \( R(u,v) \) by \( \sqrt{1 - u^2 - v^2} \), summing all of them and then multiplying the result by \( du \ dv/4\pi \). As a note, avoid using values of \( u \) and \( v \) that are exactly on the unit circle since \( R(u,v)/\sqrt{1 - u^2 - v^2} \) will be infinite at these points. In my code, I set \( R(u,v)/\sqrt{1 - u^2 - v^2} \) to zero if \( \sqrt{1 - u^2 - v^2} = 0 \).

As a practical word of caution, the above computation of \( \bar{R} \) will be accurate only if \( du \) and \( dv \) are small. This, in turn, requires that we use vary large FFTs to compute the antenna pattern. Even with modern computers, this can lead to memory overflow problems.