Appendix 4A: Change of Variable Formula in a Double Integral

Consider the transformation

\[ z = g(x, y) \]  \hspace{1cm} (4A.1)

\[ w = h(x, y) \]

between the \((x,y)\) plane and the \((z,w)\) plane. Assume that (4A.1) has the inverse

\[ x = \phi(z, w) \]  \hspace{1cm} (4A.2)

\[ y = \psi(z, w). \]

Assume that (4A.1) and (4A.2) have continuous first-partial derivatives. Define the Jacobian

\[
\frac{\partial (z, w)}{\partial (x, y)} \equiv \det \begin{bmatrix}
\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}
\end{bmatrix} = \frac{\partial (g, h)}{\partial (x, y)} = \det \begin{bmatrix}
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\
\frac{\partial h}{\partial x} & \frac{\partial h}{\partial y}
\end{bmatrix}.
\]  \hspace{1cm} (4A.3)

As discussed in the class notes, this Jacobian relates incremental regions in the \((z,w)\) and \((x,y)\) planes.

Consider Regions \(S_1\) and \(S_2\) in the \((z,w)\) and \((x,y)\) planes, respectively, as abstractly depicted by Figure 4A-1. Suppose we are interested in integrating some function \(f(z,w)\) over region \(S_1\). By a change of variable, this integral can be performed over region \(S_2\). In fact, the celebrated change of variable formula of multi-dimensional Calculus states that

\[
\int_{S_1} \int f(z, w) \, dz \, dw = \int_{S_2} \int f(g(x, y), h(x, y)) \left| \frac{\partial (g, h)}{\partial (x, y)} \right| \, dx \, dy. \]  \hspace{1cm} (4A.4)
As the examples that follow show, integration over $S_2$ can be easier than integration over $S_1$.

**Example 4A-1:** In rectangular $x$-$y$ coordinates, a circle of radius $r$ is described by the equation $x^2 + y^2 = r^2$. Find the area of this circle. In rectangular coordinates, the area is computed as

$$\text{Area} = \int_{-r}^{r} \int_{-(r^2-x^2)^{1/2}}^{(r^2-x^2)^{1/2}} dy \, dx,$$  \hspace{1cm} (4A.5)

a result that can be integrated in closed form. A simplification can be achieved by changing to polar coordinates and exploiting the obvious circular symmetry. We use the transformation

$$\rho = \sqrt{x^2 + y^2}, \quad \rho \geq 0,$$

$$\theta = \tan^{-1}(y/x), \quad -\pi < \theta \leq \pi,$$  \hspace{1cm} (4A.6)

which has the inverse

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta.$$  \hspace{1cm} (4A.7)

The Jacobian is
\[
\frac{\partial(x, y)}{\partial(\rho, \theta)} = \det \begin{bmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{bmatrix} = \rho. \quad (4A.8)
\]

From (4A.4), we can evaluate

\[
\text{Area} = \int_{-\rho}^{\rho} \int_{-(\rho^2 - x^2)^{1/2}}^{(\rho^2 - x^2)^{1/2}} dy dx = \int_{-\pi}^{\pi} \int_{0}^{\rho} \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| dp d\theta = 2\pi \int_{0}^{\rho} \rho \, dp = \pi \rho^2. \quad (4A.9)
\]

**Example 4A-2:** Several important problems in probability theory involve an integral of the form

\[
\int_{-T}^{T} \int_{-T}^{T} g(x - y) \, dx \, dy,
\]

where the integrand depends only on the difference \( x - y \) (not absolute \( x \) and/or \( y \)). Often, this integral can be simplified by the transformation

\[
u = x - y
\]

\[
v = x + y
\]

with inverse

\[
x = \frac{1}{2}(u + v)
\]

\[
y = \frac{1}{2}(v - u)
\]

The Jacobian is

\[
\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}. \quad (4A.13)
\]
In the (u,v)-plane, as u goes from –2T to 0, v traverses from –2T-u to 2T+u, as can be seen from examination of Figure 4A-2. Also, as u goes from 0 to 2T, v traverses from –2T+u to 2T-u. Hence, the integral (4A.10) can be expressed as

$$
\int_{-T}^{T} \int_{-T}^{T} g(x-y) \, dx \, dy = \int_{-2T}^{2T} \int_{-2T}^{2T} \frac{\partial(x,y)}{\partial(u,v)} \, du \, dv \\
= \int_{-2T}^{0} \int_{-2T}^{2T+u} \frac{1}{2} g(u) \, dv \, du + \int_{0}^{2T} \int_{-2T}^{2T-u} \frac{1}{2} g(u) \, dv \, du \\
= \int_{-2T}^{0} (2T+u) g(u) \, du + \int_{0}^{2T} (2T-u) g(u) \, du \\
= \int_{-2T}^{2T} (2T-u) g(u) \, du
$$

(4A.14)

Often, the integral over u on the right-hand side of (4A.14) is easier to evaluate than the original integral in the x-y coordinate system.