Chapter 7 - Correlation Functions

Let $X(t)$ denote a random process. The autocorrelation of $X$ is defined as

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2) \, dx_1 \, dx_2. \quad (7-1)$$

The autocovariance function is defined as

$$C_X(t_1, t_2) = E[(X(t_1) - \eta_X(t_1)) (X(t_2) - \eta_X(t_2))] = R_X(t_1, t_2) - \eta_X(t_1) \eta_X(t_2), \quad (7-2)$$

and the correlation function is defined as

$$r_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sigma_X(t_1) \sigma_X(t_2)} = \frac{R_X(t_1, t_2) - \eta_X(t_1) \eta_X(t_2)}{\sigma_X(t_1) \sigma_X(t_2)}. \quad (7-3)$$

If $X(t)$ is at least wide sense stationary, then $R_X$ depends only on the time difference $\tau = t_1 - t_2$, and we write

$$R_X(\tau) = E[X(t)X(t + \tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) \, dx_1 \, dx_2. \quad (7-4)$$

Finally, if $X(t)$ is ergodic we can write

$$R_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t + \tau) \, dt. \quad (7-5)$$

Function $r_X(\tau)$ can be thought of as a “measure of statistical similarity” of $X(t)$ and $X(t+\tau)$. If $r_X(\tau_0) = 0$, the samples $X(t)$ and $X(t+\tau_0)$ are said to be uncorrelated.
Properties of Autocorrelation Functions for Real-Valued, WSS Random Processes

1. $R_X(0) = E[X(t)X(t)] = \text{Average Power}$. 

2. $R_X(\tau) = R_X(-\tau)$. The autocorrelation function of a real-valued, WSS process is even.

Proof:

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t-\tau)X(t-\tau+\tau)] \quad (\text{Due to WSS})$$

$$= R_X(-\tau) \quad (7-6)$$

3. $|R_X(\tau)| \leq R_X(0)$. The autocorrelation is maximum at the origin.

Proof:

$$E\left[(X(t) \pm X(t+\tau))^2\right] = E\left[X(t)^2 + X(t+\tau)^2 + \pm 2X(t)X(t+\tau)\right] \geq 0$$

$$= R_X(0) + R_X(0) \pm 2R_X(\tau) \geq 0 \quad (7-7)$$

Hence, $|R_X(\tau)| \leq R_X(0)$ as claimed.

4. Assume that WSS process $X$ can be represented as $X(t) = \eta + X_{ac}(t)$, where $\eta$ is a constant and $E[X_{ac}(t)] = 0$. Then,

$$R_X(\tau) = E\left[(\eta + X_{ac}(t))(\eta + X_{ac}(t+\tau))\right]$$

$$= E[\eta^2] + 2\eta E[X_{ac}(t)] + E\left[X_{ac}(t)X_{ac}(t+\tau)\right] \quad (7-8)$$

$$= \eta^2 + R_{X_{ac}}(\tau).$$

5. If each sample function of $X(t)$ has a periodic component of frequency $\omega$ then $R_X(\tau)$ will have a periodic component of frequency $\omega$. 

Updates at http://www.ece.uah.edu/courses/ee385/
Example 7-1: Consider \( X(t) = Acos(\omega t + \theta) + N(t) \), where \( A \) and \( \omega \) are constants, random variable \( \theta \) is uniformly distributed over \((0, 2\pi)\), and wide sense stationary \( N(t) \) is independent of \( \theta \) for every time \( t \). Find \( R_X(\tau) \), the autocorrelation of \( X(t) \).

\[
R_X(\tau) = E\left[ \{Acos(\omega t + \theta) + N(t)\} \{Acos(\omega (t+\tau) + \theta) + N(t+\tau)\} \right] \\
= \frac{A^2}{2} E\left[ \cos(2\omega t + \omega \tau + 2\theta) + \cos(\omega \tau) \right] + E\left[ A \cos(\omega t + \theta) N(t + \tau) \right] \\
+ E\left[ N(t) A \cos(\omega (t+\tau) + \theta) \right] + E\left[ N(t) N(t + \tau) \right] \\
= \frac{A^2}{2} \cos(\omega \tau) + R_N(\tau) \\
\tag{7-9}
\]

So, \( R_X(\tau) \) contains a component at \( \omega \), the same frequency as the periodic component in \( X \).

6. Suppose that \( X(t) \) is ergodic, has zero mean, and it has no periodic components; then

\[
\lim_{\tau \to \infty} R_X(\tau) = 0. \\
\tag{7-10}
\]

That is, \( X(t) \) and \( X(t+\tau) \) become uncorrelated for large \( \tau \).

7. Autocorrelation functions cannot have an arbitrary shape. As will be discussed in Chapter 8, for a WSS random process \( X(t) \) with autocorrelation \( R_X(\tau) \), the Fourier transform of \( R_X(\tau) \) is the \textit{power density spectrum} (or simply \textit{power spectrum}) of the random process \( X \). And, the power spectrum must be non-negative. Hence, we have the additional requirement that

\[
\mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega \tau} d\tau = \int_{-\infty}^{\infty} R_X(\tau) \cos(\omega \tau) d\tau \geq 0 \\
\tag{7-11}
\]

for all \( \omega \) (the even nature of \( R_X \) was used to obtain the right-hand side of (7-11)). Because of this, in applications, you will not find autocorrelation functions with flat tops, vertical sides, or
any jump discontinuities in amplitude (these features cause “oscillatory behavior”, and negative values, in the Fourier transform). Autocorrelation $R(\tau)$ must vary smoothly with $\tau$.

**Example 7-2: Random Binary Waveform**

Process $X(t)$ takes on only two values: $\pm A$. Every $t_a$ seconds a sample function of $X$ either “toggles” value or it remains the same (positive constant $t_a$ is known). Both possibilities are equally likely (\textit{i.e.}, $P[\text{“toggle”}] = P[\text{“no toggle”}] = 1/2$). The possible transitions occur at times $t_0 + kt_a$, where $k$ is an integer, $-\infty < k < \infty$. Time $t_0$ is a random variable that is uniformly distributed over $[0, t_a]$. Hence, given an arbitrary sample function from the ensemble, a “toggle” can occur anytime. Starting from $t = t_0$, sample functions are constant over intervals of length $t_a$, and the constant can change sign from one $t_a$ interval to the next. The value of $X(t)$ over one "$t_a$-interval" is independent of its value over any other "$t_a$-interval". Figure 7-1 depicts a typical sample function of the random binary waveform. Figure 7-2 is a timing diagram that illustrates the "$t_a$ intervals". The algorithm used to generate the process is not changing with time. As a result, it is possible to argue that the process is stationary. Also, since $+A$ and $-A$ are equally likely values for $X$ at any time $t$, it is obvious that $X(t)$ has zero mean.

![Sample function of a simple binary random process.](image_url)

**Fig. 7-1:** Sample function of a simple binary random process.
To determine the autocorrelation function $R_X(\tau)$, we must consider two basic cases.

1) Case $|\tau| > t_a$. Then, the times $t_1$ and $t_1 + \tau$ cannot be in the same "$t_a$ interval". Hence, $X(t_1)$ and $X(t_1 + \tau)$ must be independent so that

$$R(\tau) = E[X(t_1)X(t_1 + \tau)] = E[X(t_1)]E[X(t_1 + \tau)] = 0, \quad |\tau| > t_a.$$  \hspace{1cm} (7-12)

2) Case $|\tau| < t_a$. To calculate $R(\tau)$ for this case, we must first determine an expression for the probability $P[t_1$ and $t_1 + \tau$ in the same “$t_a$ interval”]. We do this in two parts: the first part is i) $0 < \tau < t_a$, and the second part is ii) $-t_a < \tau \leq 0$.

i) $0 < \tau < t_a$. Times $t_1$ and $t_1 + \tau$ may, or may not, be in the same "$t_a$-interval". However, we write

$$P[t_1$ and $t_1 + \tau$ in same "$t_a$ interval"] = $P[t_0 \leq t_1 \leq t_1 + \tau < t_0 + t_a]$$

$$= P[t_1 + \tau - t_a < t_0 \leq t_1]$$

$$= \frac{1}{t_a} [t_1 - (t_1 + \tau - t_a)]$$

$$= \frac{t_a - \tau}{t_a}, \quad 0 < \tau < t_a$$ \hspace{1cm} (7-13)

ii) $-t_a < \tau \leq 0$. Times $t_1$ and $t_1 + \tau$ may, or may not, be in the same "$t_a$-interval". However, we write
\[ P[t_1 \text{ and } t_1 + \tau \text{ in same } "t_a \text{ interval}"] = P[t_0 \leq t_1 + \tau \leq t_0 + t_a] \]
\[ = P[t_1 - t_a < t_0 \leq t_1 + \tau] \]
\[ = \frac{1}{t_a} [t_1 + \tau - (t_1 - t_a)] \]
\[ = \frac{t_a + \tau}{t_a}, \quad -t_a < \tau \leq 0 \]  

(7-14)

Combine (7-13) and (7-14), we can write

\[ P[t_1 \text{ and } t_1 + \tau \text{ in same } "t_a \text{ interval}"] = \frac{t_a - |\tau|}{t_a}, \quad |\tau| < t_a. \]  

(7-15)

Now, the product \( X(t_1)X(t_1+\tau) \) takes on only two values, plus or minus \( A^2 \). If \( t_1 \) and \( t_1 + \tau \) are in the same "\( t_a \text{-interval}" \) then \( X(t_1)X(t_1+\tau) = A^2 \). If \( t_1 \) and \( t_1 + \tau \) are in different "\( t_a \text{-intervals}" \) then \( X(t_1) \) and \( X(t_1+\tau) \) are independent, and \( X(t_1)X(t_1+\tau) = \pm A^2 \) equally likely. For \( |\tau| < t_a \) we can write

\[ R(\tau) = E[X(t_1)X(t_1+\tau)] \]
\[ = A^2 P[t_1 \text{ and } t_1 + \tau \text{ in same } "t_a \text{ interval}""] \]
\[ + A^2 P[\{t_1 \text{ and } t_1 + \tau \text{ in different } "t_a \text{ intervals}"\}, X(t_1)X(t_1+\tau) = A^2]. \]  

(7-16)

However, the last two terms on the right-hand side of (7-16) cancel out (read again the two sentences after (7-15)). Hence, we can write

\[ R(\tau) = A^2 P[t_1 \text{ and } t_1 + \tau \text{ in same } "t_a \text{ interval}""]. \]  

(7-17)
Finally, substitute (7-15) into (7-17) to obtain

$$R(\tau) = \mathbb{E}[X(t_1)X(t_1 + \tau)] = \begin{cases} A^2 \left[ \frac{t_a - |\tau|}{t_a} \right], & |\tau| < t_a \\ 0, & |\tau| > t_a \end{cases}.$$  

Equation (7-18) provides a formula for $R(\tau)$ for the random binary signal described by Figure 7-1. A plot of this formula for $R(\tau)$ is given by Figure 7-3.

**Poisson Random Points Review**

The topic of random Poisson points is discussed in Chapters 1, 2 and Appendix 9B. Let $n(t_1, t_2)$ denote the number of Poisson points in the time interval $(t_1, t_2)$. Then, these points are distributed in a Poisson manner with

$$
P[n(t_1, t_2) = k] = e^{-\lambda \tau} \frac{(\lambda \tau)^k}{k!},$$

where $\tau \equiv |t_1 - t_2|$, and $\lambda > 0$ is a known parameter. That is, $n(t_1,t_2)$ is Poisson distributed with parameter $\lambda \tau$. Note that $n(t_1,t_2)$ is an integer valued random variable with
\[ E[n(t_1, t_2)] = \lambda |t_1 - t_2| \]
\[ \text{VAR}[n(t_1, t_2)] = \lambda |t_1 - t_2| \]
\[ E[n^2(t_1, t_2)] = \text{VAR}[n(t_1, t_2)] + (E[n(t_1, t_2)])^2 = \lambda |t_1 - t_2| + \lambda^2 |t_1 - t_2|^2. \]  

Note that \( E[n(t_1, t_2)] \) and \( \text{VAR}[n(t_1, t_2)] \) are the same, an unusual result for random quantities. If \((t_1, t_2)\) and \((t_3, t_4)\) are non-overlapping, then the random variables \( n(t_1, t_2) \) and \( n(t_3, t_4) \) are independent. Finally, constant \( \lambda \) is the average point density. That is, \( \lambda \) represents the average number of points in a unit length interval.

**Poisson Random Process**

Define the Poisson random process

\[ X(t) = \begin{cases} n(0,t), & t > 0 \\ -n(0,t), & t < 0 \end{cases} \]  

A typical sample function is illustrated by Figure 7-4.

**Mean of Poisson Process**

For any fixed \( t \geq 0 \), \( X(t) \) is a Poisson random variable with parameter \( \lambda t \). Hence,

\[ E[X(t)] = \lambda t, \ t \geq 0. \]  

The time varying nature of the mean implies that process \( X(t) \) is nonstationary.

**Fig. 7-4:** Typical sample function of Poisson random process.
Autocorrelation of Poisson Process

The autocorrelation is defined as \( R(t_1, t_2) = E[X(t_1)X(t_2)] \) for \( t_1 \geq 0 \) and \( t_2 \geq 0 \). First, note that

\[
R(t, t) = \lambda |t| + \lambda^2 t^2, \quad -\infty < t < \infty, \tag{7-23}
\]
a result obtained from the known 2\textsuperscript{nd} moment of a Poisson random variable. Next, we show that

\[
R(t_1, t_2) = \begin{cases} 
\lambda t_2 + \lambda^2 t_1 t_2 & \text{for } 0 < t_2 < t_1 \\
\lambda t_1 + \lambda^2 t_1 t_2 & \text{for } 0 < t_1 < t_2 
\end{cases} \tag{7-24}
\]

**Proof: case \( 0 < t_1 < t_2 \)**

We consider the case \( 0 < t_1 < t_2 \). The random variables \( X(t_1) \) and \( \{X(t_2) - X(t_1)\} \) are independent since they are for non-overlapping time intervals. Also, \( X(t_1) \) has mean \( \lambda t_1 \), and \( \{X(t_2) - X(t_1)\} \) has mean \( \lambda(t_2 - t_1) \). As a result,

\[
E[X(t_1)\{X(t_2) - X(t_1)\}] = E[X(t_1)]E[X(t_2) - X(t_1)]
\]

\[
= \lambda t_1 \cdot \lambda(t_2 - t_1) \tag{7-25}
\]

Use this result to obtain

\[
R(t_1, t_2) = E[X(t_1)X(t_2)] = E[X(t_1)\{X(t_1)+X(t_2) - X(t_1)\}]
\]

\[
= E[X^2(t_1)] + E[X(t_1)\{X(t_2) - X(t_1)\}] \tag{7-26}
\]

\[
= \lambda |t_1| + \lambda^2 t_1^2 + \lambda t_1 \cdot \lambda(t_2 - t_1)
\]

\[
= \lambda |t_1| + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_1 < t_2
\]
Case $0 < t_2 < t_1$ is similar to the case shown above. Hence, for the Poisson process, the autocorrelation function is

$$R(t_1, t_2) = \lambda t_2 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_2 < t_1$$

$$= \lambda t_1 + \lambda^2 t_1 t_2 \quad \text{for } 0 < t_1 < t_2$$

(7-27)

Semi-Random Telegraph Signal

The Semi-Random Telegraph Signal is defined as

$$X(0) = 1$$

$$X(t) = 1 \text{ if number of Poisson Points in (0,t) is even}$$

$$= -1 \text{ if number of Poisson Points in (0,t) is odd}$$

(7-28)

for $-\infty < t < \infty$. Figure 7-5 depicts a typical sample function of this process. In what follows, we find the mean and autocorrelation of the semi-random telegraph signal.

First, note that

$$P[X(t) = 1] = P[\text{even number of pts in (0, t)]}$$

$$= P[0 \text{ pts in (0,t)] + P[2 \text{ pts in (0,t)]} + P[4 \text{ pts in (0,t)]} + \cdots$$

(7-29)

$$= e^{-\lambda |t|} \left(1 + \frac{\lambda^2 |t|^2}{2!} + \frac{\lambda^4 |t|^4}{4!} + \cdots \right) = e^{-\lambda |t|} \cosh(\lambda |t|).$$

Fig. 7-5: A typical sample function of the semi-random telegraph signal.
Note that (7-29) is valid for $t < 0$ since it uses $|t|$. In a similar manner, we can write

\[
P[X(t) = -1] = P[\text{odd number of pts in } (0, t)] \\
= P[1 \text{ pts in } (0,t)] + P[3 \text{ pts in } (0,t)] + P[5 \text{ pts in } (0,t)] + \cdots \\
= e^{-\lambda |t|} \left( \lambda |t| + \frac{\lambda^3 |t|^3}{3!} + \frac{\lambda^5 |t|^5}{5!} + \cdots \right) = e^{-\lambda |t|} \sinh(\lambda |t|).
\]

As a result of (7-29) and (7-30), the mean is

\[
E[X(t)] = +1 \times P[X(t) = +1] - 1 \times P[X(t) = -1] \\
= e^{-\lambda |t|} \left( \cosh(\lambda |t|) - \sinh(\lambda |t|) \right) \\
= e^{-2\lambda |t|}.
\]

The constraint $X(0) = 1$ causes a nonzero mean that dies out with time. Note that $X(t)$ is not WSS since its mean is time varying.

Now, find the autocorrelation $R(t_1, t_2)$. First, suppose that $t_1 - t_2 \equiv \tau > 0$, and $-\infty < t_2 < \infty$. If there is an even number of points in $(t_2, t_1)$, then $X(t_1)$ and $X(t_2)$ have the same sign and

\[
P[X(t_1) = 1, X(t_2) = 1] = P[X(t_1) = 1 | X(t_2) = 1] P[X(t_2) = 1] \\
= \{\exp[-\lambda \tau] \cosh(\lambda \tau) \} \{\exp[-\lambda |t_2|] \cosh(\lambda |t_2|)\}
\]

\[
P[X(t_1) = -1, X(t_2) = -1] = P[X(t_1) = -1 | X(t_2) = -1] P[X(t_2) = -1] \\
= \{\exp[-\lambda \tau] \cosh(\lambda \tau) \} \{\exp[-\lambda |t_2|] \sinh(\lambda |t_2|)\}
\]

for $t_1 - t_2 \equiv \tau > 0$, and $-\infty < t_2 < \infty$. If there are an odd number of points in $(t_2, t_1)$, then $X(t_1)$ and
X(t_2) have different signs, and we have

\[ P[X(t_1) = 1, X(t_2) = -1] = P[X(t_1) = 1 | X(t_2) = -1] P[X(t_2) = -1] \]
\[ = \{\exp[-\lambda \tau] \sinh(\lambda \tau)\} \{\exp[-\lambda \tau/2] \sinh(\lambda \tau/2)\} \]  

(7-34)

\[ P[X(t_1) = -1, X(t_2) = 1] = P[X(t_1) = -1 | X(t_2) = 1] P[X(t_2) = 1] \]
\[ = \{\exp[-\lambda \tau] \sinh(\lambda \tau)\} \{\exp[-\lambda \tau/2] \cosh(\lambda \tau/2)\} \]  

(7-35)

for \( t_1 - t_2 \equiv \tau > 0 \), and \(-\infty < t_2 < \infty\). The product \( X(t_1)X(t_2) \) is +1 with probability given by the sum of (7-32) and (7-33); it is -1 with probability given by the sum of (7-34) and (7-35). Hence, its expected value can be expressed as

\[ R(t_1, t_2) = E[X(t_1)X(t_2)] \]
\[ = e^{-\lambda \tau} \cosh(\lambda \tau) \left[ e^{-\lambda \tau/2} \{\cosh(\lambda \tau/2) + \sinh(\lambda \tau/2)\} \right] \]
\[ -e^{-\lambda \tau} \sinh(\lambda \tau) \left[ e^{-\lambda \tau/2} \{\cosh(\lambda \tau/2) + \sinh(\lambda \tau/2)\} \right]. \]

(7-36)

Using standard identities, this result can be simplified to produce

\[ R(t_1, t_2) = E[X(t_1)X(t_2)] \]
\[ = e^{-\lambda \tau} \left[ \cosh(\lambda \tau) - \sinh(\lambda \tau) \right] e^{-\lambda \tau/2} \left[ \cosh(\lambda \tau/2) + \sinh(\lambda \tau/2) \right] \]
\[ = e^{-\lambda \tau} \left[ e^{-\lambda \tau} \right] e^{-\lambda \tau/2} \left[ e^{\lambda \tau/2} \right] \]
\[ = e^{-2\lambda \tau}, \quad \text{for } \tau = t_1 - t_2 > 0. \]

(7-37)
Due to symmetry (the autocorrelation function must be even), we can conclude that

\[ R(t_1, t_2) = R(\tau) = e^{-2\lambda |\tau|}, \quad \tau = |t_1 - t_2|, \quad (7-38) \]

is the autocorrelation function of the semi-random telegraph signal, a result illustrated by Fig. 7-6. Again, note that the semi-random telegraph signal is not WSS since it has a time-varying mean.

**Random Telegraph Signal**

Let \( X(t) \) denote the semi-random telegraph signal discussed above. Consider the process \( Y(t) = \alpha X(t) \), where \( \alpha \) is a random variable that is independent of \( X(t) \) for all time. Furthermore, assume that \( \alpha \) takes on only two values: \( \alpha = +1 \) and \( \alpha = -1 \) equally likely. Then the mean of \( Y \) is \( E[Y] = E[\alpha X] = E[\alpha]E[X] = 0 \) for all time. Also, \( R_Y(\tau) = E[\alpha^2]R_X(\tau) = R_X(\tau) \), a result depicted by Figure 7-6. \( Y \) is called the *Random Telegraph Signal* since it is “entirely random” for all time.

![Autocorrelation function for both the semi-random and random telegraph signals.](http://www.ece.uah.edu/courses/ee385/)

**Fig. 7-6**: Autocorrelation function for both the semi-random and random telegraph signals.
t. Note that the Random Telegraph Signal is WSS.

**Autocorrelation of Wiener Process**

Consider the Wiener process that was introduced in Chapter 6. If we assume that $X(0) = 0$ (in many textbooks, this is part of the definition of a Wiener process), then the autocorrelation of the Wiener process is $R_X(t_1, t_2) = 2D\{\min(t_1, t_2)\}$. To see this, first recall that a Wiener process has independent increments. That is, if $(t_1, t_2)$ and $(t_3, t_4)$ are non-overlapping intervals (i.e., $0 \leq t_1 < t_2 \leq t_3 < t_4$), then increment $X(t_2) - X(t_1)$ is statistically independent of increment $X(t_4) - X(t_3)$.

Now, consider the case $t_1 > t_2 \geq 0$ and write

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[(X(t_1) - X(t_2) + X(t_2))X(t_2)]$$

$$= E[(X(t_1) - X(t_2))X(t_2)] + E[X(t_2)]X(t_2)$$

$$= 0 + 2D t_2.$$ (7-39)

By symmetry, we can conclude that

$$R_X(t_1, t_2) = 2D\{\min(t_1, t_2)\}, \quad t_1 & t_2 \geq 0,$$ (7-40)

for the Wiener process $X(t)$, $t \geq 0$, with $X(0) = 0$.

**Correlation Time**

Let $X(t)$ be a zero mean (i.e., $E[X(t)] = 0$) W.S.S. random process. The *correlation time* of $X(t)$ is defined as

$$\tau_X \equiv \frac{1}{R_X(0)} \int_0^\infty |R_X(\tau)| d\tau.$$ (7-41)

Intuitively, time $\tau_X$ gives some measure of the time interval over which “significant” correlation exists between two samples of process $X(t)$.
For example, consider the random telegraph signal described above. For this process the correlation time is

\[ \tau_x = \frac{1}{\lambda} \int_0^{\infty} e^{-2\lambda \tau} \, d\tau = \frac{1}{2\lambda}. \] (7-42)

In Chapter 8, we will relate correlation time to the spectral bandwidth (to be defined in Chapter 8) of a W.S.S. process.

**Crosscorrelation Functions**

Let \( X(t) \) and \( Y(t) \) denote real-valued random processes. The *crosscorrelation* of \( X \) and \( Y \) is defined as

\[ R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y; t_1, t_2) \, dx \, dy. \] (7-43)

The *crosscovariance* function is defined as

\[ C_{XY}(t_1, t_2) = E[(X(t_1) - \eta_X(t_1))(Y(t_2) - \eta_Y(t_2))] = R_{XY}(t_1, t_2) - \eta_X(t_1)\eta_Y(t_2) \] (7-44)

Let \( X(t) \) and \( Y(t) \) be WSS random processes. Then \( X(t) \) and \( Y(t) \) are said to be *jointly stationary in the wide sense* if \( R_{XY}(t_1, t_2) = R_{XY}(\tau), \tau = t_1 - t_2 \). For jointly stationary in the wide sense processes the crosscorrelation is

\[ R_{XY}(\tau) = E[X(t + \tau)Y(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y; \tau) \, dx \, dy \] (7-45)

**Warning:** Some authors define \( R_{XY}(\tau) = E[X(t)Y(t+\tau)] \); in the literature, there is controversy in the definition of \( R_{XY} \) over which function is shifted. For \( R_{XY} \), the order of the subscript is significant! In general, \( R_{XY} \neq R_{YX} \).
For jointly stationary random processes X and Y, we show some elementary properties of the cross correlation function.

1. $R_{YX}(\tau) = R_{XY}(-\tau)$. To see this, note that

$$R_{YX}(\tau) = E[Y(t+\tau)X(t)] = E[Y(t-\tau+\tau)X(t-\tau)] = E[Y(t)X(t-\tau)] = R_{XY}(-\tau). \quad (7-46)$$

2. $R_{YX}(\tau)$ does not necessarily have its maximum at $\tau = 0$; the maximum can occur anywhere. However, we can say that

$$2|R_{XY}(\tau)| \leq R_X(0) + R_Y(0). \quad (7-47)$$

To see this, note that

$$E[\{X(t+\tau)\pm Y(t)\}^2] = E[X^2(t+\tau)] \pm 2E[X(t+\tau)Y(t)] + E[Y^2(t)] \geq 0$$

$$= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0. \quad (7-48)$$

Hence, we have $2|R_{XY}(\tau)| \leq R_X(0) + R_Y(0)$ as claimed.

**Linear, Time-Invariant Systems: Expected Value of the Output**

Consider a linear time invariant system with impulse response $h(t)$. Given input $X(t)$, the output $Y(t)$ can be computed as

$$Y(t) = L[X(t)] = \int_{-\infty}^{\infty} X(\tau)h(t-\tau) \, d\tau \quad (7-49)$$

The notation $L[\bullet]$ denotes a linear operator (the convolution operator in this case). As given by (7-49), output $Y(t)$ depends only on input $X(t)$, initial conditions play no role here (assume that all initial conditions are zero).
Convolution and expectation are integral operators. In applications that employ these operations, it is assumed that we can interchange the order of convolution and expectation. Hence, we can write

\[ E[Y(t)] = E[L[X(t)]] = E\left[ \int_{-\infty}^{\infty} X(\tau) h(t - \tau) \, d\tau \right] \]

\[ = \int_{-\infty}^{\infty} E[X(\tau)] h(t - \tau) \, d\tau \]

\[ = \int_{-\infty}^{\infty} \eta_X(\tau) h(t - \tau) \, d\tau. \quad (7-50) \]

More generally, in applications, it is assumed that we can interchange the operations of expectation and integration so that

\[ E\left[ \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n \right] = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} E[f(t_1, \ldots, t_n)] \, dt_1 \cdots dt_n, \quad (7-51) \]

for example (f is a random function involving variables \( t_1, \ldots, t_n \)).

As a special case, assume that input \( X(t) \) is wide-sense stationary with mean \( \eta_x \). Equation (7-50) leads to

\[ \eta_Y = E[Y(t)] = \eta_X \left[ \int_{-\infty}^{\infty} h(t - \tau) \, d\tau \right] = \eta_X \left[ \int_{-\infty}^{\infty} h(\tau) \, d\tau \right] = \eta_X H(0), \quad (7-52) \]

where \( H(0) \) is the DC response (i.e., DC gain) of the system.

**Linear, Time-Invariant Systems: Input/Output Cross Correlation**

Let \( R_X(t_1, t_2) \) denote the autocorrelation of input random process \( X(t) \). We desire to find \( R_{XY}(t_1, t_2) = E[x(t_1)y(t_2)] \), the crosscorrelation between input \( X(t) \) and output \( Y(t) \) of a linear, time-invariant system.
Theorem 7-1

The cross correlation between input $X(t)$ and output $Y(t)$ can be calculated as (both $X$ and $Y$ are assumed to be real-valued)

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = L_2\left[R_X(t_1, t_2)\right] = \int_{-\infty}^{\infty} R_X(t_1, t_2 - \alpha) h(\alpha) d\alpha$$  \hspace{1cm} (7-53)

Notation: $L_2[\cdot]$ means operate on the $t_2$ variable (the second variable) and treat $t_1$ (the first variable) as a fixed parameter.

Proof: In (7-53), the convolution involves folding and shifting the “$t_2$ slot” so we write

$$Y(t_2) = \int_{-\infty}^{\infty} X(t_2 - \alpha) h(\alpha) d\alpha \implies X(t_1)Y(t_2) = \int_{-\infty}^{\infty} X(t_1)X(t_2 - \alpha) h(\alpha) d\alpha,$$  \hspace{1cm} (7-54)

a result that can be used to derive

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} E[X(t_1)X(t_2 - \alpha)] h(\alpha) d\alpha$$

$$= \int_{-\infty}^{\infty} R_X(t_1, t_2 - \alpha) h(\alpha) d\alpha$$  \hspace{1cm} (7-55)

$$= L_2\left[R_X(t_1, t_2)\right].$$

Special Case: Theorem 7.1 for WSS $X(t)$

Suppose that input process $X(t)$ is WSS. Let $\tau = t_1 - t_2$ and write (7-55) as

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} X(\tau + \alpha) h(\alpha) d\alpha = \int_{-\infty}^{\infty} R_X(\tau - \alpha) h(-\alpha) d\alpha = R_X(\tau) * h(\tau)$$  \hspace{1cm} (7-56)

Note that $X(t)$ and $Y(t)$ are jointly wide sense stationary.
Theorem 7-2

The autocorrelation $R_Y(t_1,t_2)$ can be obtained from the crosscorrelation $R_{XY}(t_1,t_2)$ by the formula

$$R_Y(t_1,t_2) = L_1[R_{XY}(t_1,t_2)] = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2) h(\alpha) \, d\alpha \quad (7-57)$$

Notation: $L_1[\cdot]$ means operate on the $t_1$ variable (the first variable) and treat $t_2$ (the second variable) as a fixed parameter.

Proof: In (7-57), the convolution involves folding and shifting the “$t_1$ slot” so we write

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1-\alpha) h(\alpha) \, d\alpha \implies Y(t_1)Y(t_2) = \int_{-\infty}^{\infty} Y(t_2)X(t_1-\alpha) h(\alpha) \, d\alpha \quad (7-58)$$

Now, take the expected value of (7-58) to obtain

$$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = \int_{-\infty}^{\infty} E[Y(t_2)X(t_1-\alpha)] h(\alpha) \, d\alpha = \int_{-\infty}^{\infty} R_{XY}(t_1-\alpha,t_2) h(\alpha) \, d\alpha = L_1[R_{XY}(t_1,t_2)] , \quad (7-59)$$

a result that completes the proof of our theorem.

Special Case: Theorem 7.2 for WSS $X(t)$

Suppose that input process $X(t)$ is WSS. Then, as seen by (7-56), $X$ and $Y$ are jointly WSS so that (7-59) becomes

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty} R_{XY}(t_1-t_2-\alpha) h(\alpha) \, d\alpha . \quad (7-60)$$

Let $\tau = t_1 - t_2$ and write (7-60) as
\( R_Y(\tau) = R_{XY}(\tau) * h(\tau). \)  \( \quad (7-61) \)

Theorems 7.1 and 7.2 can be combined into a single formula for finding \( R_Y(t_1, t_2) \) in terms of \( R_X(t_1, t_2) \). Simply substitute (7-55) into (7-59) to obtain

\[
R_Y(t_1, t_2) = \int_{-\infty}^{\infty} R_{XY}(t_1 - \alpha, t_2) h(\alpha) \, d\alpha = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\beta) \, d\beta \right] h(\alpha) \, d\alpha \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_1 - \alpha, t_2 - \beta) h(\alpha)h(\beta) \, d\alpha \, d\beta, \quad (7-62)
\]

an important "double convolution" formula for \( R_Y(t_1, t_2) \) in terms of \( R_X(t_1, t_2) \). For WSS input \( X(t) \), the output autocorrelation can be written as

\[
R_Y(\tau) = R_{XY}(\tau) * h(\tau) = R_X(\tau) * \{h(-\tau) * h(\tau)\}, \quad (7-63)
\]

a result obtained by substituting (7-56) into (7-61). Figure 7-7 illustrates this fundamental result. Note that WSS input \( X(t) \) produce WSS output \( Y(t) \).

**Example 7-3**

A zero-mean, stationary process \( X(t) \) with autocorrelation \( R_x(\tau) = q \delta(\tau) \) (white noise) is applied to a linear system with impulse response \( h(t) = e^{-ct} U(t), c > 0 \). Find \( R_{xy}(\tau) \) and \( R_y(\tau) \) for this system. Since \( X \) is WSS, we know that

\[
R_{XY}(\tau) = E[X(t + \tau)Y(t)] = R_X(\tau) * h(-\tau) = q \delta(\tau) * e^{ct}U(-\tau) = q e^{ct}U(-\tau). \quad (7-64)
\]

\[
\begin{array}{ccc}
R_X(\tau) & \rightarrow & h(\tau)h(-\tau) \\
& & R_Y(\tau) = [h(\tau)h(-\tau)]R_X(\tau)
\end{array}
\]

**Fig. 7-7:** Output autocorrelation in terms of input autocorrelation for the WSS input case.
Note that X and Y are jointly WSS. That $R_{XY}(\tau) = 0$ for $\tau > 0$ should be intuitive since X(t) is a white noise process. Now, the autocorrelation of Y can be computed as

$$R_Y(\tau) = E[Y(t+\tau)Y(t)] = R_X(\tau) *[h(\tau)*h(-\tau)] = [R_X(\tau)*h(\tau)] * h(\tau)$$

$$= R_{XY}(\tau) * h(\tau) = \{q e^{c\tau}U(-\tau)\} * \{e^{-c\tau}U(\tau)\}$$

$$= \frac{q}{2c} e^{-\frac{1}{2}c|\tau|}, \quad -\infty < \tau < \infty. \quad (7-65)$$

Note that output Y(t) is not “white” noise; samples of Y(t) are correlated with each other. Basically, system h(t) filtered white-noise input X(t) to produce an output Y(t) that is correlated. As we will see in Chapter 8, input X(t) is modeled as having an infinite bandwidth; the system “bandlimited” its input to form an output Y(t) that has a finite bandwidth.

**Example 7-4 (from Papoulis, 3rd Ed., pp. 311-312)**

A zero-mean, stationary process X(t) with autocorrelation $R_x(\tau) = q\delta(\tau)$ (white noise) is applied at $t = 0$ to a linear system with impulse response $h(t) = e^{-ct}U(t)$, $c > 0$. See Figure 7-8. Assume that the system is “at rest” initially (the initial conditions are zero so that $Y(t) = 0$, $t < 0$). Find $R_{xy}$ and $R_y$ for this system.

In a subtle way, this problem differs from the previous example. Since the input is applied at $t = 0$, the system “sees” a non-stationary input. Hence, we must analyze the general, nonstationary case. As shown below, processes X(t) and Y(t) are not jointly wide sense stationary, and Y(t) is not wide sense stationary. Also, it should be obvious that $R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = 0$ for $t_1 < 0$ or $t_2 < 0$ (note how this differs from Example 7-3).

![Figure 7-8: System with random input applied at t = 0.](http://www.ece.uah.edu/courses/ee385/)

Updates at http://www.ece.uah.edu/courses/ee385/
R_{xy}(t_1, t_2) equals the response of the system to \( R_x(t_1-t_2) = q\delta(t_1-t_2) \), when \( t_1 \) is held fixed and \( t_2 \) is the independent variable (the input \( \delta \) function occurs at \( t_2 = t_1 \)). For \( t_1 > 0 \) and \( t_2 > 0 \), we can write

\[
R_{xy}(t_1, t_2) = L_2 \left[ R_X(t_1, t_2) \right] = \int_{-\infty}^{\infty} R_X(t_1, t_2 - \alpha) h(\alpha) \, d\alpha
\]

\[
= \int_{-\infty}^{\infty} q \delta(t_1 - t_2 + \alpha) h(\alpha) \, d\alpha = q h(-[t_1 - t_2])
\]

\[
= q \exp[-c(t_2 - t_1)] U(t_2 - t_1), \quad t_1 > 0, \quad t_2 > 0,
\]

\[
= 0, \quad \text{if} \quad t_1 < 0 \text{ or } t_2 < 0,
\]

a result that is illustrated by Figure 7-9. For \( t_2 < t_1 \), output \( Y(t_2) \) is uncorrelated with input \( X(t_1) \), as expected (this should be intuitive). Also, for \( (t_2 - t_1) > 5/c \) we can assume that \( X(t_1) \) and \( Y(t_2) \) are uncorrelated. Finally, note that \( X(t) \) and \( Y(t) \) are not jointly wide sense stationary since \( R_{XY} \) depends on absolute \( t_1 \) and \( t_2 \) (and not only the difference \( \tau = t_1 - t_2 \)).

Now, find the autocorrelation of the output; there are two cases. The first case is \( t_2 > t_1 > 0 \) for which we can write
Y(t) is not stationary. The reason for this is simple (and intuitive). Input X(t) is applied at t = 0, and the system is “at rest” before this time (Y(t) = 0, t < 0). For a few time constants, this fact is “remembered” by the system (the system “has memory”). For t1 and t2 larger than 5 time constants (t1, t2 > 5/(2c)), “steady state” can be assumed, and the output autocorrelation can be approximated as

\[ R_Y(t_1, t_2) \approx \frac{q}{2c} e^{-c|t_2-t_1|}. \]  

(7-69)
Example 7-5: Let $X(t)$ be a real-valued, WSS process with autocorrelation $R(\tau)$. For any fixed $T > 0$, define the random variable

$$S_T \equiv \int_{-T}^{T} X(t) dt.$$  \hspace{1cm} (7-70)

Express the second moment $\mathbb{E}[S_T^2]$ as a single integral involving $R(\tau)$. First, note that

$$S_T^2 \equiv \int_{-T}^{T} X(t_1) dt_1 \int_{-T}^{T} X(t_2) dt_2 = \int_{-T}^{T} \int_{-T}^{T} X(t_1)X(t_2) dt_1 dt_2,$$  \hspace{1cm} (7-71)

a result that leads to

$$\mathbb{E} \left[ S_T^2 \right] = \int_{-T}^{T} \int_{-T}^{T} \mathbb{E} \left[ X(t_1)X(t_2) \right] dt_1 dt_2 = \int_{-T}^{T} \int_{-T}^{T} R(t_1-t_2) dt_1 dt_2.$$  \hspace{1cm} (7-72)

The integrand in (7-72) depends only on one quantity, namely the difference $t_1 - t_2$. Therefore, Equation (7-72) should be expressible in terms of a single integral in the variable $\tau \equiv t_1 - t_2$. To see this, use $\tau \equiv t_1 - t_2$ and $\tau' = t_1 + t_2$ and map the $(t_1, t_2)$ plane to the $(\tau, \tau')$ plane (this relationship has an inverse), as shown by Fig. 7-10. As discussed in Appendix 4A, the integral (7-72) can be expressed as

$$\int_{-T}^{T} \int_{-T}^{T} R(t_1-t_2) dt_1 dt_2 = \int_{\mathcal{R}_2} R(\tau) \left| \frac{\partial(t_1, t_2)}{\partial(\tau, \tau')} \right| d\tau d\tau',$$  \hspace{1cm} (7-73)

where $\mathcal{R}_2$ is the “rotated square” region in the $(\tau, \tau')$ plane shown on the right-hand side of Fig. 7-10. For use in (7-73), the Jacobian of the transformation is
In the $(\tau, \tau')$-plane, as $\tau$ goes from $-2T$ to 0, the quantity $\tau'$ traverses from $-2T-\tau$ to $2T+\tau$, as can be seen from examination of Fig. 7-10. Also, as $\tau$ goes from 0 to $2T$, the quantity $\tau'$ traverses from $-2T+\tau$ to $2T-\tau$. Hence, we have

$$E\left[ S_t^2 \right] = \int_{-T}^{T} \int_{-T}^{T} R(t_1 - t_2) dt_1 dt_2 = \int_{-2T}^{0} \int_{-(2T+\tau)}^{2T+\tau} \frac{1}{2} R(\tau) d\tau' d\tau + \int_{0}^{2T} \int_{-(2T-\tau)}^{2T-\tau} \frac{1}{2} R(\tau) d\tau' d\tau$$

$$= \int_{-2T}^{0} (2T+\tau) R(\tau) d\tau + \int_{0}^{2T} (2T-\tau) R(\tau) d\tau$$

$$= \int_{-2T}^{2T} (2T-|\tau|) R(\tau) d\tau$$

a single integral that can be evaluate given autocorrelation function $R(\tau)$. 

Fig. 7-10: Geometry used to change variables from $(t_1, t_2)$ to $(\tau, \tau')$ in the double integral that appears in Example 7-5.