Chapter 14: Markov Process Dynamic System State Model

There are many applications of dynamic systems driven by stationary Gaussian noise. In these applications, the system is modeled by a differential equation with a Gaussian noise forcing function. If the system is linear, the system state vector is Gaussian, and its mean and covariance matrix can be found easily (numerical methods may be necessary in the case of linear time varying systems). If the system is nonlinear, the system state vector is not generally Gaussian, and modeling/analyzing the system becomes more difficult. This chapter introduces theory and techniques that are useful in modeling/analyzing linear/nonlinear systems driven by Gaussian noise.

Often, the bandwidth of a noise forcing function is large compared to the bandwidth of the system driven by the noise. To the system, the noise forcing function “looks white”; its spectral density “looks flat” over the system bandwidth, even thought it does not have an infinite bandwidth/power. Under these circumstances, it is common to model the noise forcing function as white Gaussian noise. In general, this modeling assumption (known as the diffusion approximation in the literature) simplifies system analysis, and it allows the problem at hand to be treated by a vast body of existing knowledge.

All lumped-parameter dynamic systems (i.e., systems that can be modeled by a differential equation), be they linear or nonlinear, time-varying or time-invariant, have an important feature in common. Assuming white Gaussian noise excitation, all lumped-parameter dynamic systems have a state vector that can be modeled as a Markov process. Roughly speaking, what this means is simple: Given the system state at time $t_0$ and the input noise for $t \geq t_0$, one can determine the system state for $t \geq t_0$. Future values of the system state can be determined using the present value of the system state and the input noise; past values of the state are not necessary.

For a lumped-parameter dynamic system driven by white Gaussian noise, we are interested in determining the probability density function that describes the system state vector. In general, this density function evolves with time (the system state is a nonstationary process), starting from some known density at $t = 0$. As discussed in this chapter, the desired density function satisfies a
partial differential equation known as the *Fokker-Planck* equation.

This chapter is devoted to laying the foundation for the analysis of this Markov state model. First, from Chapter 6 of these class notes, the classical random walk is reviewed; it is a simple example of a discrete Markov process. As step size and the time between successive steps approach zero, the random walk approaches the Wiener process, a simple continuous-time Markov process. The Wiener process is described by a probability density function that satisfies the diffusion equation, a simple example of a Fokker-Planck equation. After discussing this simple example, a more general first-order system model is introduced, and the Fokker-Planck equation is developed that describes the model.

**The Random Walk - A Simple Markov Process**

Suppose a man takes a random walk on a straight-line path; he starts his walk $m$ steps to the right of the origin. With probability $p$ (alternatively, $q \equiv 1 - p$), he takes a step to the right (alternatively, left). Suppose that each step is of length $l$ meters, and each step is completed in $\tau$ seconds. After $N$ steps (completed in $N\tau$ seconds), the man is located $X_d(N)$ steps from the origin; note that $-N + m \leq X_d(N) \leq N + m$ since the man starts at $m$ steps to the right of the origin. If $X_d(N)$ is positive (negative), the man is located to the right (left) of the origin.

The quantity $P[X_d(N) = n \mid X_d(0) = m]$ denotes the probability that the man's location is $n$ steps to the right of the origin, after $N$ steps, given that he starts at $m$ steps to the right of the origin. The calculation of this probability is simplified greatly by the assumption, implied in the previous paragraph, that the man takes independent steps. That is, the direction taken at the $N^{th}$ step is independent of $X_d(k)$, $0 \leq k \leq N - 1$, and the directions taken at all previous steps. Also simplifying the development is the assumption that probability $p$ does not depend on step index $N$.

A formula for $P[X_d(N) = n \mid X_d(0) = m]$ is developed in Chapter 6 of these class notes; this development is summarized here. Let $v \equiv n - m$, so that $v$ denotes the man's net increase in the number of steps to the right after he has completed $N$ steps. Also, $R_{mn}$ (alternatively, $L_{mn}$) denotes the number of steps to the right (alternatively, left) that are required if the man starts and finishes $m$ and $n$, respectively, steps from the origin. Then, it is easily seen that
\[ R_{nm} = \frac{N + \nu}{2} \]
\[ L_{nm} = \frac{N - \nu}{2} \]  

(14-1)

if \( |\nu| \leq N \) and \( N + \nu, N - \nu \) are even; otherwise, integers \( R_{nm} \) and \( L_{nm} \) do not exist. In terms of these integers, the desired result is

\[ P[X_d(N) = n \mid X_d(0) = m] = P[R_{nm} \text{ steps to the right out of } N \text{ steps}] \]
\[ = \frac{N!}{R_{nm}! L_{nm}!} p^{R_{nm}} q^{L_{nm}} \]  

(14-2)

if integers \( R_{nm} \) and \( L_{nm} \) exist, and

\[ P[X_d(N) = n \mid X_d(0) = m] = 0 \]  

(14-3)

if \( R_{nm} \) and \( L_{nm} \) do not exist.

For \( Npq \gg 1 \), an asymptotic approximation is available for (14-2). In the development that follows, it is assumed that \( p = q = 1/2 \). According to the DeMoivre-Laplace, for \( N/4 \gg 1 \) and \( |R_{nm} - N/2| < \sqrt{N/4} \), the approximation

\[ P[X_d(N) = n \mid X_d(0) = m] \approx \frac{N!}{R_{nm}! L_{nm}!} \left( \frac{1}{2} \right)^{R_{nm}} \left( \frac{1}{2} \right)^{L_{nm}} \exp \left[-\frac{(R_{nm} - N/2)^2}{2(N/4)}\right] \]  

(14-4)

can be made.

**Limit of the Random Walk - the Wiener Process**

Recall that each step corresponds to a distance of \( \ell \) meters, and each step is completed in \( \tau_s \) seconds. At time \( t = N\tau_s \), let \( X(N\tau_s) \) denote the man's physical displacement from the origin.
Then $X(N\tau_s)$ is a random process given by $X(N\tau_s) \equiv \ell X_d(N)$, since $X_d(N)$ denotes the number of steps the man is from the origin after he takes $N$ steps. Note that $X(N\tau_s)$ is a discrete-time random process that takes on only discrete values.

For large $N$ and small $\ell$ and $\tau_s$, the probabilistic nature of $X(N\tau_s)$ is of interest. First, note that $P[X(N\tau_s) = \ell | X(0) = m] = P[X_d(N) = n | X_d(0) = m]$; this observation and the Binomial distribution function leads to the result

$$P[X(N\tau_s) \leq \ell | X(0) = m] = P[\text{Number of Steps to Right} \leq R_{nm}]$$

$$= \sum_{k=0}^{R_{nm}} \binom{N}{k} \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{N-k}.$$  \hspace{1cm} (14-5)

For large $N$, the DeMoivre-Laplace leads to the approximation

$$P[X(N\tau_s) \leq \ell | X(0) = m] = \mathcal{G} \left( \frac{R_{nm} - N/2}{\sqrt{N/4}} \right) = \mathcal{G} \left( \frac{\nu}{\sqrt{N}} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\nu/\sqrt{N}} \exp \left[ -\frac{1}{2} u^2 \right] du.$$  \hspace{1cm} (14-6)

where $\mathcal{G}$ is the distribution function for a zero-mean, unit-variance Gaussian random variable.

The discrete random walk process outlined above has the continuous Wiener process as a formal limit. To see this, let $\ell \to 0$, $\tau_s \to 0$ and $N \to \infty$ in such a manner that

$$\frac{\ell^2}{2\tau_s} \to D$$

$$t = N \tau_s$$

$$x = n \ell$$  \hspace{1cm} (14-7)

$$x_0 = m \ell$$

$$X(t) = X(N\tau_s).$$
where D is known as the \textit{diffusion constant}. In terms of D, x, x_0 and t, the results of (14-7) can be used to write

$$\frac{\nu}{\sqrt{N}} = \frac{(x-x_0)/\ell}{\sqrt{t/\tau_s}} = \frac{1}{\sqrt{2D}} \frac{x-x_0}{\sqrt{t}}.$$  \hfill (14-8)

The probabilistic nature of the limiting form of \(X(t)\) is seen from (14-6) and (14-8). In the limit, the process \(X(t)\) is described by the first-order conditional distribution function

$$F(x; t \mid x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-x_0)/\sqrt{2Dt}} \exp\left[-\frac{1}{2} u^2\right] du$$  \hfill (14-9)

and the first-order conditional density function

$$f(x, t \mid x_0) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right].$$  \hfill (14-10)

When \(X(0) = x_0 = 0\), this result describes the conditional probability density function of a continuous-time Wiener process. Clearly, process \(X(t)\) is Gaussian, and it is nonstationary since it has a variance that grows with time. \(X(t)\) is continuous, with probability one, but its sample functions are nowhere differentiable. It is a simple example of a \textit{diffusion process}.

\textbf{The Diffusion Equation For the Transition Density Function}

As discussed in Chapter 6 of these notes, the conditional density (14-10) satisfies the one-dimensional \textit{diffusion equation}

$$\frac{\partial}{\partial t} f(x, t \mid x_0) = D \frac{\partial^2}{\partial x^2} f(x, t \mid x_0)$$  \hfill (14-11)
with initial condition

\[ f(x, t|x_0) \big|_{t = 0} = \delta(x - x_0) \]  \hspace{1cm} (14-12)

and boundary condition

\[ f(x, t|x_0) \big|_{x = \pm \infty} = 0. \]  \hspace{1cm} (14-13)

Initial condition (14-12) means that process \( X \) starts at \( x_0 \). Boundary condition (14-13) implies that probability cannot accumulate at infinity; often, (14-13) is referred to as *natural boundary conditions*.

Diffusion equation (14-11) describes how probability diffuses (or flows) with time. To draw this analogy, note that \( f \) describes the density of probability (or density of probability particles) on the one-dimensional real line. That is, \( f \) can be assigned units of particles/meter. Since \( D \) has units of meters\(^2\)/second, a unit check on both sides of (14-11) produces

\[
\left( \frac{1}{\text{second}} \right) \left( \frac{\text{particles}}{\text{meter}} \right) = \left( \frac{\text{meter}^2}{\text{second}} \right) \left( \frac{1}{\text{meter}} \right) \left( \frac{\text{particles}}{\text{meter}} \right)^2.
\]  \hspace{1cm} (14-14)

Diffusion phenomenon is a transport mechanism that describes flow in many important applications (heat, electric current, molecular, etc.).

Equation (14-11) implies that probability is conserved in much the same way that the well-know continuity equation implies the conservation of electric charge. Write (14-11) as

\[
\frac{\partial}{\partial t} f = -\nabla \mathcal{F},
\]  \hspace{1cm} (14-15)

where
\[ \mathcal{I} = -D \frac{\partial}{\partial x} f , \quad (14-16) \]

and \( \nabla \) is the divergence operator. The quantity \( \mathcal{I} \) is a one-dimensional probability current, and it has units of particles/second. Note the similarity between (14-15) and the well-known continuity equation for electrical charge.

Probability current \( \mathcal{I}(x,t \mid x_0) \) indicates the rate of particle flow past point \( x \) at time \( t \). Let \( (x_1, x_2) \) denote an interval; integrate (14-15) over this interval to obtain

\[ \frac{\partial}{\partial t} P[x_1 < X(t) \leq x_2 \mid x_0] = \frac{\partial}{\partial t} \int_{x_1}^{x_2} f(x,t \mid x_0) \, dx = -[\mathcal{I}(x_2,t \mid x_0) - \mathcal{I}(x_1,t \mid x_0)]. \quad (14-17) \]

As illustrated by Fig. 14-1, the left-hand side of this equation represents the time rate of probability build-up on \( (x_1, x_2) \). That is, between the limits of \( x_1 \) and \( x_2 \), the area under \( f \) is changing at a rate equal to the left-hand side of (14-17). As depicted by Fig. 14-1, the right-hand side of (14-17) represents the probability currents entering the ends of the interval \( (x_1, x_2) \).

**An Absorbing Boundary On the Random Walk**

The quantity \( X_d(N) \) is unconstrained in the discrete random walk discussed so far. Now, consider placing an absorbing boundary at \( n_1 \). No further displacements are possible after the man reaches the boundary at \( n_1 \); the man stops his random walk the instant he arrives at the
boundary (he is absorbed). Let $X_A(N)$ take the place of $X_d(N)$ to distinguish the fact that an absorbing boundary exists at $n_1$. That is, after taking $N$ steps, the man is $X_A(N)$ steps to the right of the origin, given an absorbing boundary at $n_1$. Clearly, we have

$$X_A(N) = X_d(N) \quad \text{if } X_d(n) < n_1 \text{ for all } n \leq N$$

$$= n_1 \quad \text{if } X_d(n) = n_1 \text{ for some } n \leq N$$

(14-18)

An absorbing boundary has applications in many problems of practical importance.

As before, assume that the man starts his random walk at $m$ steps from the origin where $m < n_1$. This initial condition implies that $X_A(0) = m$ since random process $X_A$ denotes the man's displacement (in steps) from the origin. He takes random steps; either he completes $N$ of them, or he is absorbed at the boundary before completing $N$ steps. For the random walk with an absorbing boundary, the quantity $P[n, N \mid m; n_1]$ denotes the probability that $X_A(N) = n$ given that $X_A(0) = m$ and an absorbing boundary exists at $n_1$. In what follows, an expression is developed for this probability.

It is helpful to trace the man's movement by using a plane as shown by Figures 14-2a and 14-2b. On these diagrams, the horizontal axis denotes displacement, in steps, from the origin; the vertical axis denotes the total number of steps taken by the man. Every time the man takes a step, he moves upward on the diagram; also, he moves laterally to the right or left. The absorbing boundary is depicted on these figures by a solid vertical line at $n_1$. In the remainder of this section, these diagrams are used to illustrate the reflection principle for dealing with random processes that hit absorbing boundaries.

Figure 14-2a depicts two $N$-step sequences (the solid line paths) that start at $m$ and arrive at $n$. One of these is "forbidden" since it intersects the boundary. A "forbidden" $N$-step sequence is one that intersects the boundary one or more times. For the present argument, assume that a "forbidden" sequence is not stopped (or altered in any way) by the boundary. For all steps above the last point of contact with the boundary, the "forbidden" sequence on Fig. 14-2a has been
reflected across the boundary to produce a dashed-line path that leads to the point \(2n_1 - n\), the reflected (across the boundary) image of the point \(n\). In this same manner, every "forbidden" path that starts at \(m\) and reaches \(n\) can be partially reflected to produce a unique path that leads to the image point \(2n_1 - n\).

The solid line path on Fig. 14-2b is an N-step sequence that arrives at the point \(2n_1 - n\). As was the case on Fig. 14-2a, point \(2n_1 - n\) is the mirror image across the boundary of point \(n\). For all steps above the last point of contact with the boundary, the solid-line sequence on Fig. 14-2b has been reflected across the boundary to produce a dashed-line path that leads to the point \(n\) (we have mapped the solid-line path that reaches the image point into a “forbidden” sequence that reaches point \(n\)). In this same manner, every path that reaches image point \(2n_1 - n\) can be partially reflected to produce a unique “forbidden” path that leads to \(n\).

From the observations outlined in the last two paragraphs, it can be concluded that a one-to-one correspondence exists between N-step "forbidden" sequences that reach point \(n\) and N-step sequences that reach the image point \(2n_1 - n\). That is, for every "forbidden" sequence that reaches \(n\), there is a sequence that reaches \(2n_1 - n\). And, for every sequence that reaches \(2n_1 - n\) there is a “forbidden” sequence that reaches \(n\). Out of all N-step sequences that start at \(m\), the proportion that are “forbidden” and reach \(n\) is exactly equal to the proportion that reach the image.
point. This observation is crucial in the development of \( P[n, N \mid m; n_1] \).

Without the boundary in place, the computation of \( P[n, N \mid m] \) involves computing the relative frequency of the man arriving at \( X_d = n \) after \( N \) steps. That is, to compute the probability \( P[n, N \mid m] \), the number of \( N \) step sequences that leave \( m \) and lead to \( n \) must be normalized by the total number of distinct \( N \) step sequences that leave \( m \). \( P[n, N \mid m] \) can be represented as

\[
P[n, N \mid m] = \frac{\# \text{N step sequences that leave } m \text{ and reach } n}{\text{Total number of N step sequences that leave } m}.
\] (14-19)

With the boundary at \( n_1 \) in place, the computation of \( P[n, N \mid m; n_1] \) involves computing the relative frequency of the man arriving at \( X_A = n \) after \( N \) steps. To compute \( P[n, N \mid m; n_1] \) when the boundary is in place, a formula similar to (14-19) can be used, but the number of "forbidden" sequences (i.e., those that would otherwise be absorbed at the boundary) that reach \( n \) must be subtracted from the total number (i.e., the number without a boundary) of \( N \)-step sequences that lead to \( n \). That is, when the boundary is in place, (14-19) must be modified to produce

\[
P[n, N \mid m; n_1] = \frac{\# \text{N step sequences that leave } m \text{ and reach } n \text{ with no boundary in place}}{- \# \text{N step "forbidden" sequences that leave } m \text{ and reach } n \text{ without a boundary in place}}{\text{Total number of N step sequences that leave } m}.
\] (14-20)

But the number of \( N \) step “forbidden” sequences that reach \( n \) is exactly equal to the number of sequences that reach the image point \( 2n_1 - n \). Hence, (14-20) can be modified to produce

\[
P[n, N \mid m; n_1] = \frac{\# \text{N step sequences that leave } m \text{ and reach } n \text{ with no boundary in place}}{- \# \text{N step sequences that leave } m \text{ and reach } 2n_1 - n \text{ without a boundary in place}}{\text{Total number of N step sequences that leave } m}.
\] (14-21)
This equation is rewritten as

\[
P[n, N \mid m; n] = P[n, N \mid m] - P[2n_1 - n, N \mid m],
\]

(14-22)

where \( P[n, N \mid m] \) is given by (14-2). For the absorbing boundary case, the probability of reaching \( n \) can be expressed in terms of probabilities that are calculated for the boundary-free case.

**An Absorbing Boundary On the Wiener Process**

As before, suppose that each step corresponds to a distance of \( \ell \) meters, and it takes \( \tau_s \) seconds to take a step. Furthermore, \( X_A(N\tau_s) = \ell X_A(N) \) denotes the man's physical distance (in meters) from the origin. Also, for the case where an absorbing boundary exists at \( \ell n_i > \ell m \), \( P[\ell n, N\tau_s \mid \ell m; \ell n_i] \) denotes the probability that the man is \( \ell n \) meters from the origin at \( t = N\tau_s \), given that he starts at \( \ell m \) when \( t = 0 \). Using the argument which led to (14-22), it is possible to write

\[
P[\ell n, N\tau_s \mid \ell m; \ell n_1] = P[\ell n, N\tau_s \mid \ell m] - P[2n_1 \ell - n \ell, N\tau_s \mid \ell m].
\]

(14-23)

The argument that lead to (14-10) can be applied to (14-23), and a density \( f_A(x, t \mid x_0; x_1) \) that describes the limit process \( X_A(t) \) can be obtained. As \( \ell \to 0 \), \( \tau_s \to 0 \) and \( N \to \infty \) in the manner described by (14-7), the limiting argument that produced (14-10) can be applied to (14-23); the result of taking the limit is

\[
f_A(x, t \mid x_0; x_1) = \frac{1}{\sqrt{4\pi Dt}} \left[ \exp\left( -\frac{(x-x_0)^2}{4Dt} \right) - \exp\left( -\frac{(x_1-x-x_0)^2}{4Dt} \right) \right], \quad x < x_1,
\]

(14-24)

where \( \ell n_i \to x_1 \) is the location of the absorbing boundary. For \( x < x_1 \), density \( f_A(x, t \mid x_0; x_1) \) is described by (14-24). As discussed next, this density contains a delta function at \( x = x_1 \), to account for the portion of sample functions that have been absorbed by time \( t \).
For \( x < x_1 \), density \( f_A(x, t \mid x_0; x_1) \) is equal to the right-hand-side of (14-24). At \( x = x_1 \), density \( f_A(x, t \mid x_0; x_1) \) must contain a delta function of time-varying weight

\[
w(t \mid x_0, x_1) \equiv 1 - \int_{-\infty}^{x_1} f_A(x, t \mid x_0; x_1) \, dx,
\]

the probability that the process has been absorbed by the boundary (at \( x_1 \)) sometime during the interval \([0, t]\). Of course, \( f_A(x, t \mid x_0; x_1) = 0 \) for \( x > x_1 \). Figure 14-3 depicts density \( f_A(x, t \mid x_0; x_1) \) for \( 2Dt = 1, x_0 = 0 \) and \( x_1 = 1 \).

From (14-24), note that

\[
\lim_{x \to x_1} f_A(x, t \mid x_0; x_1) = 0 \quad (14-26)
\]

for all \( t \geq 0 \). That is, the density function vanishes as the boundary is approached from the left. If \( X_A(t) \) represents the position of a random particle, then (14-26) implies that, only rarely, can the particle be found in the vicinity of the boundary.

**A Reflecting Boundary On the Random Walk**

On the discrete random walk, the above-discussed boundary at \( n_1 \) can be made *reflecting.*

![Figure 14-3: \( f_A(x, t \mid x_0, x_1) \) for \( x_0 = 0, x_1 = 1 \) and \( 2Dt = 1 \).](image-url)
To distinguish the fact that the boundary at \( n_1 \) is reflecting, let \( X_R(N) \) denote the man’s displacement (number of steps to the right of the origin) after he has taken \( N \) steps. By definition, the process with a reflecting boundary is represented as

\[
X_R(N) = \begin{cases} 
X_d(N) & \text{if } X_d(N) < n_1 \\
2n_1 - X_d(N) & \text{if } X_d(N) \geq n_1
\end{cases}
\]

(14-27)

When \( X_d \) exceeds the boundary it is reflected through the boundary to form \( X_R \).

A version of (14-20) can be obtained for the case of a reflecting boundary at \( x_1 \). Again, the \( N \) step sequences that leave \( m \) and arrive at \( n \) must be tallied. Every sequence that leaves \( m \) and arrives at \( n \) without the boundary maps to a sequence that leaves \( m \) and arrives at \( n \) with the boundary in place. Also, every sequence that leaves \( m \) and arrives at \( 2n_1 - n \) without the boundary maps to a sequence that leaves \( m \) and arrives at \( n \) with the boundary in place. Hence, for the case of a reflecting boundary at \( n_1 \), we can write

\[
P[n, N|m; n_1] = \frac{\left\{ \# \text{N step sequences that leave } m \text{ and reach } n \text{ without the boundary in place} \right\} + \left\{ \# \text{N step "forbidden" sequences that leave } m \text{ and reach } n \right\}}{\text{Total number of N step sequences that leave } m}. \quad (14-28)
\]

Note the similarity between (14-28) and (14-20).

**A Reflecting Boundary On the Wiener Process**

As before, suppose that each step corresponds to a distance of \( \ell \) meters, and it takes \( \tau_s \) seconds to take a step. Furthermore, \( X_R(N\tau_s) = \ell X_R(N) \) denotes the man's physical distance (in meters) from the origin. Also, for the case of a reflecting boundary at \( \ell n_1 > \ell m \), \( P[\ell n, N\tau_s | \ell m; \ell n_1] \) denotes the probability that the man is \( \ell n \) meters from the origin at \( t = N\tau_s \), given that he
starts at $\ell m$ when $t = 0$. Using the argument which led to (14-22), it is possible to write

$$P[\ell n, N \tau_s | \ell m; \ell n_1] = P[\ell n, N \tau_s | \ell m] + P[2 n_1 \ell - n \ell, N \tau_s | \ell m].$$  \hspace{1cm} (14-29)

The argument that lead to (14-10) can be applied to (14-23), and a density $f_R(x, t | x_0, x_1)$ that describes the limit process $X_R(t)$ can be obtained. As $\ell \to 0$, $\tau_s \to 0$ and $N \to \infty$ in the manner described by (14-7), the limiting argument that produced (14-10) can be applied to (14-29); the result of taking the limit is

$$f_R(x, t | x_0; x_1) = \frac{1}{\sqrt{4\pi Dt}} \left[ \exp \left( -\frac{(x-x_0)^2}{4Dt} \right) + \exp \left( -\frac{(2x_1-x-x_0)^2}{4Dt} \right) \right], \quad x \leq x_1,$$

$$= 0, \quad x > x_1$$

where $\ell n_1 \to x_1$ is the location of the reflecting boundary. Figure 14-4 depicts a plot of $f_R(x, t | x_0, x_1)$ for the case $x_0 = 0$, $x_1 = 1$ and $2Dt = 1$.

![Figure 14-4](image-url)
The First-Order Markov Process

Usually, simplifying assumptions are made when performing an engineering analysis of a nonlinear system driven by noise. Often, assumptions are made that place limits on the amount of information that is required to analyze the system. For example, it is commonly assumed that a finite dimensional model can be used to describe the system. The model is described by a finite number of state variables which are modeled as random processes. An analysis of the system might involve the determination of the probability density function that describes these state variables. A second common assumption has to do with how this probability density evolves with time, and what kind of initial data it depends on. This assumption states that the future evolution of the density function can be expressed in terms of the present values of the state variables; knowledge of past values of the state variables is not necessary. As discussed in this chapter, this second assumption means that the state vector can be modeled as a continuous Markov process. Furthermore, the process has a density function that satisfies a parabolic partial differential equation known as the Fokker-Planck equation (also known as Kolmogorov’s Forward Equation).

The one-dimensional Markov process and Fokker-Planck equation are discussed in this section. Unlike the situation in multi-dimensional problems, a number of exact closed-form results can be obtained for the one-dimensional case, and this justifies treating the case separately. Also, the one-dimensional case is simpler to deal with from a notational standpoint.

A random process has the Markov property if its distribution function at any future instant, conditioned on present and past values of the process, does not depend on the past values. Consider increasing, but arbitrary, values of time $t_1 < t_2 < \ldots < t_n$, where $n$ is an arbitrary positive integer. A random process $X(t)$ has the Markov property if its conditional probability distribution function satisfies
for all values $x_1, x_2, \ldots, x_n$ and all sequences $t_1 < t_2 < \ldots < t_n$.

The Wiener and random walk processes discussed in Section 6.1 are examples of Markov processes. In the development that produced (14-10), the initial condition $x_0$ was specified at $t = 0$. Now, suppose the initial condition is changed so that $x_0$ is specified at $t = t_0$. In this case, transitions in the displacement random process $X$ can be described by the conditional probability density function

$$f(x, t \mid x_0, t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp \left[ -\frac{(x-x_0)^2}{4D(t-t_0)} \right]$$

for $t > t_0$. Note that the displacement random process $X$ is Markov since, for $t$ greater than $t_0$, density (14-32) can be expressed in terms of the displacement value $x_0$ at time $t_0$; prior to time $t_0$, the history of the displacement is not relevant.

If random process $X(t)$ is a Markov process, the joint density of $X(t_1), X(t_2), \ldots, X(t_n)$, where $t_1 < t_2 < \ldots < t_n$, has a simple representation. First, recall the general formula

$$f(x_n, t_n \mid x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = f(x_n, t_n \mid x_{n-1}, t_{n-1}; \ldots; x_1, t_1) f(x_{n-1}, t_{n-1} \mid x_{n-2}, t_{n-2}; \ldots; x_1, t_1) \cdots$$

$$\cdots f(x_2, t_2 \mid x_1, t_1) f(x_1, t_1)$$

Now, utilize the Markov property to write (14-33) as
Equation (14-34) states that a Markov process $X(t)$, $t \geq t_0$, is completely specified by an initial marginal density $f(x_0, t_0)$ (think of this marginal density as specifying an “initial condition” on the process) and a first-order conditional density

$$f(x, t \mid x_1, t_1), \ t \geq t_1, \ t_1 \geq t_0.$$  \hspace{4cm} (14-35)

For this reason, conditional densities of the form (14-35) are known as *transition densities*. Based on physical reasoning, it is easy to see that (14-35) should satisfy

$$f(x, t \mid x_1, t_1) = \delta(x - x_1).$$  \hspace{4cm} (14-36)

Some important special cases arise regarding the time dependence of the marginal and transitional densities of a Markov process. A Markov process is said to be *homogeneous* if $f(x_2, t_2 \mid x_1, t_1)$ is invariant to a shift in the time origin. In this case, the transition density depends only on the time difference $t_2 - t_1$. Now, recall that stationarity implies that both $f(x, t)$ and $f(x_2, t_2 \mid x_1, t_1)$ are invariant to a shift in the time origin. Hence, stationary Markov processes are homogeneous. However, the converse of this last statement is not generally true.

**An Important Application of Markov Processes**

Consider a physical problem that can be modeled as a first-order system driven by white Gaussian noise. Let $X(t)$, $t \geq t_0$, denote the state of this system; the statistical properties of state $X$ are of interest here. Suppose that the initial condition $X(t_0)$ is a random variable that is independent of the white noise driving the system. Then state $X(t)$ belongs to a special class of Markov processes known as diffusion processes. As is characteristic of diffusion processes,
almost all sample functions of $X$ are continuous, but they are differentiable nowhere. Finally, these statements are generalized easily to $n^{th}$-order systems driven by white Gaussian noise.

As an example of a first-order system driven by white Gaussian noise, consider the RL circuit illustrated by Fig. 14-5. In this circuit, current $i(t)$, $t \geq t_0$, is the state, and white Gaussian noise $v_{in}(t)$ is assumed to have a mean of zero. Inductance $L(i)$ is a coil wound on a ferrite core with a nonlinear B-H relationship; the inductance is a known positive function of the current $i$ with derivative $dL/di$. The initial condition $i(t_0)$ is assumed to be a zero-mean, Gaussian random variable, and it is independent of the input noise $v_{in}(t)$ for all $t$.

The formal differential equation that describes state $i(t)$ is

$$\frac{d}{dt} Li = \left[ \frac{dL}{di} i + L \right] \frac{d}{dt} i = -Ri + v_{in}$$

Equation (14-37) is equivalent to

$$\frac{d}{dt} i = \frac{-Ri}{(dL/di)i + L} + \frac{v_{in}}{(dL/di)i + L}.$$  \hspace{1cm} (14-38)

However, the problem with the previous two equations is that sample functions of current $i$ are not differentiable, so Equations (14-37) and (14-38) only serve as symbolic representations of the circuit dynamic model. Now, recall from Section 6.1.5 that white noise $v_{in}$ can be represented as a generalized derivative of a Wiener process. If $W_t$ denotes this Wiener process, and $i_t \equiv i(t)$

![Figure 14-5: A simple RL circuit.](image-url)
denotes the circuit current (in these representations, the independent time variable is depicted as a subscript), then it is possible to say that

\[
di_t = -\frac{Ri_t}{(dL/di)t_L + L} 
+ \frac{dW_t}{(dL/di)t_L + L} \tag{14-39}
\]

is formally equivalent to (14-38).

Equations (14-38) and (14-39) are stochastic differential equations, and they should be considered as nothing but formal symbolic representations for the integral equation

\[
i_t - i_{k} = \int_{t_k}^{t} \frac{Ri_\tau}{(dL/di)t_L + L} \, d\tau + \int_{t_k}^{t} \frac{dW_\tau}{(dL/di)t_L + L}, \tag{14-40}
\]

where \( t > t_k \geq t_0 \). On the right-hand side of (14-40), the first integral can be interpreted in the classical Riemann sense. However, sample functions of \( W_t \) are not of bounded variation, so the second integral cannot be a Riemann-Stieltjes integral. Instead, it can be interpreted as a stochastic Itô integral, and a major field of mathematical analysis exists to support this effort.

On the right-hand side of (14-40), the stochastic differential \( dW_t \) does not depend on \( i_t, t < t_k \). Based on this observation, it is possible to conjecture that any probabilistic description of future values of \( i_t, t > t_k \), when conditioned on the present value \( i_{tk} \) and past values \( i_{tk-1}, i_{tk-2}, \ldots \), does not depend on the past current values. That is, the structure of (14-40) suggests that \( i_t \) is Markov. The proof of this conjecture is a major result in the theory of stochastic differential equations (see Chapter 9 of Arnold, *Stochastic Differential Equations: Theory and Applications*).

Of great practical interest are methods for characterizing the statistical properties of diffusion processes that represent the state of a dynamic system driven by white Gaussian noise. At least two schools of thought exist for characterizing these processes. The first espouses direct numerical simulation of the system dynamic model. The second school is adhered to here, and it
utilizes an indirect analysis based on the Fokker-Planck equation.

**The Chapman-Kolmogorov Equation**

Suppose that $X(t)$ is a random process described by the conditional density function $f(x_3, t_3 \mid x_1, t_1)$. Clearly, this density function must satisfy

$$f(x_3, t_3 \mid x_1, t_1) = \int_{-\infty}^{\infty} f(x_3, t_3; x_2, t_2 \mid x_1, t_1) \, dx_2,$$

where $t_1 < t_2 < t_3$. Now, a standard result from probability theory can be used here; substitute

$$f(x_3, t_3; x_2, t_2 \mid x_1, t_1) = f(x_3, t_3 \mid x_2, t_2; x_1, t_1) \, f(x_2, t_2 \mid x_1, t_1)$$

into (14-41) and obtain

$$f(x_3, t_3 \mid x_1, t_1) = \int_{-\infty}^{\infty} f(x_3, t_3 \mid x_2, t_2; x_1, t_1) \, f(x_2, t_2 \mid x_1, t_1) \, dx_2.$$

Equation (14-43) can be simplified if $X$ is a Markov process. By using the Markov property, this last equation can be simplified to obtain

$$f(x_3, t_3 \mid x_1, t_1) = \int_{-\infty}^{\infty} f(x_3, t_3 \mid x_2, t_2) \, f(x_2, t_2 \mid x_1, t_1) \, dx_2.$$

This is the well-known *Chapman-Kolmogorov equation* for Markov processes (it is also known as the *Smoluchowski equation*). It provides a useful formula for the transition probability from $x_1$ at time $t_1$ to $x_3$ at time $t_3$ in terms of an intermediate step $x_2$ at time $t_2$, where $t_2$ lies between $t_1$ and $t_3$. In Section 6.3, a version of (14-44) is used in the development of the N-dimensional Fokker-Planck equation.
The One-Dimensional Kramers-Moyal Expansion

As discussed in Section 6.1.1, a limiting form of the random walk is a Markov process described by the transition density (14-10). This density function satisfies the diffusion equation (14-11). These results are generalized in this section where it is shown that a first-order Markov process is described by a transition density that satisfies an equation known as the Kramers-Moyal expansion. When the Markov process is the state of a dynamic system driven by white Gaussian noise, this equation simplifies to what is known as the Fokker-Planck equation. Equation (14-11) is a simple example of a Fokker-Planck equation.

Consider the random increment

\[ \Delta X_{t_1} \equiv X(t_1 + \Delta t) - X(t_1), \]  

(14-45)

where \( \Delta t \) is a small, positive time increment. Given that \( X(t_1) = x_1 \), the conditional characteristic function \( \Theta \) of \( \Delta X_{t_1} \) is given by

\[ \Theta(\omega;x_1,t_1,\Delta t) = E[\exp(j\omega \Delta X_{t_1}) | X_{t_1} = x_1] = \int_{-\infty}^{\infty} \exp[j\omega(x - x_1)] f(x,t_1 + \Delta t | x_1,t_1). \]

(14-46)

If the Markov process is homogeneous, then \( \Theta \) depends on the time difference \( \Delta t \) but not the absolute value of \( t_1 \). The inverse of (14-46) is

\[ f(x,t_1 + \Delta t | x_1,t_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-j\omega(x - x_1)] \Theta(\omega; x_1, t_1, \Delta t) d\omega, \]

(14-47)

which is an expression for the transition density in terms of the characteristic function of the random increment. Now, use (14-47) in

\[ f(x,t_1 + \Delta t) = \int_{-\infty}^{\infty} f(x,t_1 + \Delta t; x_1,t_1) dx_1 = \int_{-\infty}^{\infty} f(x,t_1 + \Delta t | x_1,t_1) f(x_1,t_1) dx_1 \]

(14-48)
to obtain

\[
f(x, t_1 + \Delta t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-j\omega(x-x_1)] \Theta(\omega; x_1, t_1, \Delta t) \, d\omega \right] f(x_1, t_1) \, dx_1.
\]

(14-49)

The characteristic function \( \Theta \) can be expressed in terms of the moments of the process \( X \).

To obtain such a relationship for use in (14-49), expand the exponential in (14-46) and obtain

\[
\Theta(\omega; x_1, t_1, \Delta t) = E[\exp(j\omega \Delta X_{t_1}) | X(t_1) = x_1] = E\left[ \sum_{q=0}^{\infty} \frac{(j\omega)^q}{q!} (\Delta X_{t_1})^q \right] | X(t_1) = x_1
\]

\[
= \sum_{q=0}^{\infty} \frac{(j\omega)^q}{q!} m^{(q)}(x_1, t_1, \Delta t)
\]

where

\[
m^{(q)}(x_1, t_1, \Delta t) = E[ (\Delta X_{t_1})^q | X(t_1) = x_1] = E[ (X(t_1 + \Delta t) - X(t_1))^q | X(t_1) = x_1]
\]

(14-51)

is the \( q \)th conditional moment of the random increment \( \Delta X_{t_1} \).

This expansion of the characteristic function can be used in (14-49). Substitute (14-50) into (14-49) and obtain

\[
f(x, t_1 + \Delta t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-j\omega(x-x_1)] \sum_{q=0}^{\infty} \frac{(j\omega)^q}{q!} m^{(q)}(x_1, t_1, \Delta t) \, d\omega \right] f(x_1, t_1) \, dx_1
\]

\[
= \sum_{q=0}^{\infty} \frac{1}{q!} \int_{-\infty}^{\infty} \left( j\omega \right)^q \exp[-j\omega(x-x_1)] \, d\omega \int_{-\infty}^{\infty} m^{(q)}(x_1, t_1, \Delta t) f(x_1, t_1) \, dx_1
\]

(14-52)

This result can be simplified by using the identity
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega)^q \exp[-j\omega(x-x_1)]d\omega = \frac{1}{2\pi} \left( -\frac{\partial}{\partial x} \right)^q \int_{-\infty}^{\infty} \exp[-j\omega(x-x_1)]d\omega \\
= \left( -\frac{\partial}{\partial x} \right)^q \delta(x-x_1).
\]  

The use of identity (14-53) in (14-52) results in

\[
f(x, t_1+\Delta t) = \sum_{q=0}^{\infty} \frac{1}{q!} \int_{-\infty}^{\infty} \left( -\frac{\partial}{\partial x} \right)^q \delta(x-x_1) m^{(q)}(x_1, t_1, \Delta t) f(x_1, t_1) dx_1 \\
= \sum_{q=0}^{\infty} \frac{1}{q!} \left( -\frac{\partial}{\partial x} \right)^q \int_{-\infty}^{\infty} \delta(x-x_1) m^{(q)}(x_1, t_1, \Delta t) f(x_1, t_1) dx_1 \\
= f(x, t_1) + \sum_{q=1}^{\infty} \frac{1}{q!} \left( -\frac{\partial}{\partial x} \right)^q m^{(q)}(x_1, t_1, \Delta t) f(x_1, t_1)
\]  

since \( m^{(0)} = 1 \). Now, no special significance is attached to time variable \( t_1 \) in (14-54); hence, substitute \( t \) for \( t_1 \) and write

\[
f(x, t+\Delta t) - f(x, t) = \sum_{q=1}^{\infty} \frac{1}{q!} \left( -\frac{\partial}{\partial x} \right)^q m^{(q)}(x, t, \Delta t) f(x, t).
\]

Finally, divide both sides of this last result by \( \Delta t \), and let \( \Delta t \to 0 \) to obtain the formal limit

\[
\frac{\partial}{\partial t} f(x, t) = \sum_{q=1}^{\infty} \frac{1}{q!} \left( -\frac{\partial}{\partial x} \right)^q K^{(q)}(x, t) f(x, t),
\]

where
\[ K^{(q)}(x, t) \equiv \lim_{\Delta t \to 0} \frac{E[ \{X(t + \Delta t) - X(t)\]^q \mid X(t) = x]}{\Delta t}. \]  

(14-57)

q \geq 1, are called the intensity coefficients. Integer q denotes the order of the coefficient. Equation (14-56) is called the Kramers-Moyal expansion. In general, the coefficients given by (14-57) depend on time. However, the intensity coefficients are time-invariant in cases where the underlying process is homogeneous. In what follows, we assume homogeneous processes and time-independent intensity coefficients.

**The One-Dimensional Fokker-Planck Equation**

The intensity coefficients \( K^{(q)} \) vanish for \( q \geq 3 \) in applications where \( X \) is the state of a first-order system driven by white Gaussian noise (see Risken, *The Fokker-Planck Equation*, Second Edition). This means that incremental changes \( \Delta X_t \equiv [X(t + \Delta t) - X(t)] \) in the process occur slowly enough so that their third and higher-order moments vanish more rapidly than \( \Delta t \). Under these conditions, Equation (14-56) reduces to the one-dimensional Fokker-Planck equation

\[
\frac{\partial}{\partial t} f(x, t) = -\frac{\partial}{\partial x} [K^{(1)}(x) f(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [K^{(2)}(x) f(x, t)].
\]  

(14-58)

When \( K^{(q)} = 0 \) for \( q \geq 3 \), random process \( X \) is known as a diffusion process, and its sample functions are continuous (see Risken, *The Fokker-Planck Equation*, Second Edition). Apart from initial and boundary conditions, Equation (14-58) shows that the probability density function for a diffusion process is determined completely by only first and second-order moments of the process increments.

As a simple example, consider the RL circuit depicted by Fig.14-5, where the inductance is a constant \( L \) independent of current \( i_t \). This circuit is driven by \( v_{in} \), a zero-mean, white Gaussian noise process with a double-sided spectral density of \( N_0/2 \) watts/Hz. This white noise process is
the formal derivative of a Wiener process $W_t$; the variance of an increment of this Wiener process is $(N_0/2)\Delta t$. The RL circuit is described by (14-39) and (14-40) which can be used to write

$$\Delta i_t = -\left(\frac{R}{L} i_t\right) \Delta t + \frac{1}{L} \int_t^{t+\Delta t} dW_t$$  \hspace{1cm} (14-59)

The commonly used notation $i_t \equiv i(t)$ and $\Delta i_t \equiv i(t + \Delta t) - i(t)$ is used in (14-59), and the differential $dW_t$ is formally equivalent to $v_{in} dt$. This current increment can be used in (14-57) to obtain

$$K^{(1)} = \lim_{\Delta t \to 0} \frac{\text{E}[\Delta i_t | i_t = i]}{\Delta t} = -\frac{R}{L} i$$  \hspace{1cm} (14-60)

In a similar manner, the second intensity coefficient is

$$K^{(2)} = \lim_{\Delta t \to 0} \frac{\text{E}[(\Delta i_t)^2 | i_t = i]}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{L^2} \int_t^{t+\Delta t} \int_t^{t+\Delta t} \text{E}[v_{in}(t_1) v_{in}(t_2)] dt_1 dt_2 \Delta t$$

$$= \frac{1}{L^2} \frac{N_0}{2}. \hspace{1cm} (14-61)$$

Hence, the Fokker-Planck equation that describes the RL circuit is

$$\frac{\partial f}{\partial t} = \frac{R}{L} \frac{\partial}{\partial i} (i f) + \frac{N_0}{4L^2} \frac{\partial^2}{\partial i^2} f. \hspace{1cm} (14-62)$$

Finally, as can be shown by direct substitution, a steady-state solution of this equation is

$$f(i) = \frac{1}{\sqrt{2\pi(N_0/4RL)}} \exp\left[-\frac{1}{2} \frac{i^2}{(N_0/4RL)}\right]. \hspace{1cm} (14-63)$$
The intensity coefficients used in (14-58) can be given physical interpretations when process \( X(t) \) denotes the time-dependent displacement of a particle. Consider a particle on one of an ensemble of paths (sample functions) that pass through point \( x \) at time \( t \). As the particle passes through point \( x \) at time \( t \), its velocity is dependent on which path it is on. Inspection of (14-57) shows that \( K^{(1)}(x) \) is the average of these velocities. In a similar manner, coefficient \( K^{(2)} \) can be given a physical interpretation. In \( \Delta t \) seconds after time \( t \), the particle has undergone a displacement of \( \Delta X_t = X(t+\Delta t) - X(t) \) from point \( x \). Of course, \( \Delta X_t \) depends on which path the particle is on, so there is uncertainty in the magnitude of \( \Delta X_t \). That is, after leaving point \( x \), there is uncertainty in how far the process has gone during the time increment \( \Delta t \). For small \( \Delta t \), a measure of this uncertainty is given by \( \Delta t K^{(2)}(x) \), a first-order-in-\( \Delta t \) approximation for the variance of the displacement increment \( \Delta X_t \).

In many applications, \( K^{(2)} \) is constant (independent of \( x \)). If \( K^{(2)}(x) \) is not constant, then (14-58) can be transformed into a new Fokker-Planck equation where the new coefficient \( \tilde{K}^{(2)} \) is a positive constant (for details of the transformation see pg. 97 of Risken, *The Fokker-Planck Equation*, Second Edition). For this reason, coefficient \( K^{(2)} \) in (14-58) is assumed to be a positive constant in what follows.

Note that (14-58) can be written as

\[
\frac{\partial}{\partial t} f(x, t) = -\frac{\partial}{\partial x} \left[ K^{(1)}(x) f(x, t) - \frac{1}{2} \frac{\partial}{\partial x} [K^{(2)} f(x, t)] \right] \\
= -\frac{\partial}{\partial x} \left[ K^{(1)}(x) - \frac{1}{2} \frac{\partial}{\partial x} K^{(2)} \right] f(x, t) \\
= -\nabla \cdot \mathcal{I}(x, t),
\]

where

\[
\mathcal{I}(x, t) = \left[ K^{(1)}(x) - \frac{1}{2} \frac{\partial}{\partial x} K^{(2)} \right] f(x, t),
\]
and \( \nabla \cdot \mathbf{S} \) denotes the divergence of \( \mathbf{S} \). Notice the similarity of (14-64) with the well-known continuity equation

\[
-\frac{\partial}{\partial t} \rho = \nabla \cdot \mathbf{J}
\]  

(14-66)

from electromagnetic theory. In this analogy, \( f \) (particles/meter) and \( \mathbf{S} \) (particles/second) in (14-64) are analogous to one-dimensional charge density \( \rho \) (electrons/meter) and one-dimensional current \( \mathbf{J} \) (electrons/second), respectively, in (14-66). \( \mathbf{S} \) is the probability current; in an abstract sense, it can be thought of as describing the "amount" of probability crossing a point \( x \) in the positive direction per unit time. In the literature, it is common to see \( \mathbf{S} \) cited as a flow rate of "probability particles". That is, \( \mathbf{S}(x,t) \) can be thought of as the rate of particle flow at point \( x \) and time \( t \) (see both Section 4.3 of Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1 and Section 5.2 of Gardiner, *Handbook of Stochastic Methods*).

The \( K^{(1)} f \) term in \( \mathbf{S} \) is analogous to a drift current component. To see the current aspect, recall that \( K^{(1)} \) has units of velocity if \( X \) is analogous to particle displacement, and \( f \) has units of particles per meter. Hence, the product \( K^{(1)} f \) is analogous to a one-dimensional current since

\[
\text{meters/second)(particles/meter) = particles/second}.
\]  

(14-67)

It is a drift current (i.e., a current that results from an external force or potential acting on particles) since, in applications, \( K^{(1)} \) is due to the presence of an external force. This external force acts on the particles, and

\[
\mathbf{S}_s \equiv K^{(1)} f
\]  

(14-68)

can be thought of as a drift current that results from movement of the forced particles.
In fact, drift coefficient \( K^{(1)} \) is used in

\[
U_p(x) = -2\int^x K^{(1)}(\alpha) \, d\alpha \quad (14-69)
\]

to define the one-dimensional potential function for (14-64). An important conceptual role can be developed for \( U_p \); it is more likely for probability (probability particles) to flow in the direction of a lower potential.

Figure 14-6 depicts a potential function \( U_p(x) \) which is encountered in the first-order PLL (see Stensby, *Phase-Locked Loops: Theory and Applications*) and other applications. The significant feature of this potential function is the sequence of periodically spaced potential wells. A particle can move from one well to the next, and it is more likely to move to a well of lower potential than to a well of higher potential. In the phase-locked loop, noise-induced cycle slips are associated with the movement of particles between the wells.

The component

\[
\Im d = -\frac{1}{2} \frac{\partial}{\partial x} [ K^{(2)} f(x, t) ] \quad (14-70)
\]
in (14-65) is analogous to a diffusion current (diffusion currents result in nature from a non-zero gradient of charged particles which undergo random motion - see pg. 339 of Smith and Dorf, *Devices and Systems, 5th* edition). As in (14-67), \( \mathcal{I}_d \) is analogous to a current since it has units of particles/second. That \( \mathcal{I}_d \) is analogous to a diffusion current is easy to see when \( K^{(2)} \) is a constant. In this case, \( \mathcal{I}_d \) is proportional to the gradient of the analogous charge density \( f \), and \( K^{(2)} \) is the constant diffusion coefficient. The negative sign associated with (14-70) is due to the fact that particles tend to diffuse in the direction of lower probability concentrations.

Initial and boundary conditions on \( f(x, t) \) must be supplied when a solution is sought for the Fokker-Planck equation. An initial condition is a constraint that \( f(x, t) \) must satisfy at some instant of time; initial condition \( f(x, t_1) \), where \( t_1 \) is the initial time, is specified as a function of the variable \( x \). A boundary condition is a constraint that \( f(x, t) \) must satisfy at some displacement \( x_1 \) (\( x_1 \) may be infinite); boundary condition \( f(x_1, t) \) is specified as a function of the variable \( t \). Usually, initial and boundary conditions are determined by the physical properties of the application under consideration.

**Transition Density Function**

Often, the transition density \( f(x, t \mid x_1, t_1) \) is of interest. This transition density satisfies the Fokker-Planck equation. To see this, substitute \( f(x, t) = f(x, t \mid x_1, t_1)f(x_1, t_1) \) into (14-58), cancel out the common \( f(x_1, t_1) \) term, and write

\[
\frac{\partial}{\partial t} f(x, t \mid x_1, t_1) = -\frac{\partial}{\partial x} [ K^{(1)}(x) f(x, t \mid x_1, t_1)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [ K^{(2)}(x) f(x, t \mid x_1, t_1)]
\]

(14-71)

So, the transition density \( f(x, t \mid x_1, t_1) \) for the Markov process \( X(t) \) can be found by solving (14-71) subject to

\[
f(x, t_1 \mid x_1, t_1) = \delta(x - x_1).
\]

(14-72)
In computing the transition density, the boundary conditions that must be imposed on (14-71) are problem dependent.

**Kolmogorov’s Equations**

Fokker-Planck equation (14-71) is also known as *Kolmogorov’s Forward Equation*. The adjective “Forward” refers to the fact that the equation is in terms of the temporal and spatial variables t and x, the “forward” variables (in \( f(x, t \mid x_1, t_1) \)), x and t are referred to as “forward” variables while \( x_1 \) and \( t_1 \) are called the “backward” variables).

Yes (you may be wondering), there is a *Kolmogorov Backward Equation*. In the transition density \( f(x, t \mid x_1, t_1) \), one can think of holding x, t fixed and using \( x_1, t_1 \) as the independent variables. In the “backward” variables \( x_1 \) and \( t_1 \), \( f(x, t \mid x_1, t_1) \) must satisfy

\[
\frac{\partial}{\partial t_1} f(x,t \mid x_1, t_1) = -K^{(1)}(x) \frac{\partial}{\partial x_1} f(x,t \mid x_1, t_1) - \frac{1}{2} K^{(2)}(x) \frac{\partial^2}{\partial x_1^2} f(x,t \mid x_1, t_1) , \quad (14-73)
\]

a result known as *Kolmogorov’s Backward Equation*. It is possible to show that the “backward” equation is the formal adjoint of the “forward” equation. Finally, the “backward” equation is used when studying “exit”, and “first-passage-time” problems.

**Natural, Periodic and Absorbing Boundary Conditions**

Boundary conditions must be specified when looking for a solution of (14-58). In general, the specification of boundary conditions can be a complicated task that requires the consideration of subtle issues. Fortunately, only three types of simple boundary conditions are required for most applications in the area of communication and control system analysis.

The first type of boundary conditions to be considered arise naturally in many applications where sample functions of \( X(t) \) are unconstrained in value. In these applications, the Fokker-Planck equation applies over the whole real line, and the boundary conditions specify what happens as x approaches \( \pm \infty \). With respect to x, integrate Equation (14-64) over the real line to obtain
\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x, t) \, dx = \lim_{x \to \infty} \mathcal{F}(x, t) - \lim_{x \to -\infty} \mathcal{F}(x, t). \quad (14-74) \]

Now, combine this result with the normalization condition

\[ \int_{-\infty}^{\infty} f(x, t) \, dx = 1, \quad (14-75) \]

which must hold for all t, to obtain

\[ \lim_{x \to \infty} \mathcal{F}(x, t) = \lim_{x \to -\infty} \mathcal{F}(x, t). \quad (14-76) \]

While this must be true in general, the stronger requirement

\[ \lim_{x \to \infty} \mathcal{F}(x, t) = \lim_{x \to -\infty} \mathcal{F}(x, t) = 0 \quad (14-77) \]

holds in all applications of the theory to physical problems (where probability build-up at infinity cannot occur). Furthermore, the requirement

\[ \lim_{x \to \infty} f(x, t) = \lim_{x \to -\infty} f(x, t) = 0 \quad (14-78) \]

is necessary for the normalization condition (14-75) to hold. As \( x \to \pm \infty \), the density function must satisfy the requirement that \( f(x, t) \to 0 \) on the order of \( 1/x^{1+\varepsilon} \), for some \( \varepsilon > 0 \); this requirement is necessary for convergence of the integral in (14-75). Equations (14-77) and (14-78) constitute what is called a set of natural boundary conditions.

Boundary conditions of a periodic nature are used in the analysis of phase-locked loops and other applications. They require a constraint of the form
\[ f(x, t) = f(x + L_0, t) \]
\[ \mathcal{S}(x, t) = \mathcal{S}(x + L_0, t) , \]  

(14-79)

where \( L_0 \) is the period. These are called periodic boundary conditions, and they are used for certain types of analyses when the intensity coefficients \( K^{(1)} \) and \( K^{(2)} \) are periodic functions of \( x \). Periodic intensity coefficients occur in a class of problems generally referred to as Brownian motion in a periodic potential (see Chapter 11 of Risken, *The Fokker-Planck Equation*, Second Edition).

The last type of boundary conditions discussed here are of the absorbing type. Suppose that \( X(t) \) denotes the displacement of a particle that starts at \( X(0) = x_0 \). As is illustrated by Fig. 14-7, the particle undergoes a random displacement \( X(t) \) until, at \( t = t_a \), it makes first contact with the boundary at \( x_b > x_0 \). The particle is absorbed (it vanishes) at this point of first contact. Clearly, the time interval \([0, t_a]\) from start to absorption depends on the path (sample function) taken by the particle, and the length of this time interval is a random variable.

For the case illustrated by Fig. 14-7, an absorbing boundary at \( x_b \) requires that

\[ \lim_{x \to x_b} f(x, t) = 0 \]  

(14-80)

**Figure 14-7:** At time \( t = t_a \), process \( X(t) \) hits an absorbing boundary placed at \( x = x_b \).
for all \( t \). Intuitively, this condition says that the particle can be found only rarely in a small neighborhood \((x_b - \Delta x, x_b), \Delta x > 0\), of the boundary. Previously, the boundary condition (14-80) was shown to hold for a Wiener process subjected to an absorbing boundary (see (14-26)). In the remainder of this section, this boundary condition is supported by an intuitive argument based on arriving at a contradiction; that is, (14-80) is assumed to be false, and it is shown that this leads to a contradiction (see also pg. 219 of Cox and Miller, *The Theory of Stochastic Processes*). The argument given requires that \( X(t) \) be homogeneous; however, the boundary condition holds in the more general nonhomogeneous case.

Suppose that (14-80) is not true; suppose some time interval \((t_1, t_2)\) and some displacement interval \((x_b - \Delta x, x_b)\) exist such that

\[
f(x, t) \geq \varepsilon > 0, \quad t_1 < t < t_2, \quad x_b - \Delta x < x < x_b,
\]

for some small \( \varepsilon > 0 \). That is, suppose the density \( f(x,t) \) is bounded away from zero for \( x \) in some small neighborhood of the boundary and for \( t \) in some time interval. Then, on any infinitesimal time interval \((t, t + \Delta t) \subset (t_1, t_2)\), the probability \( g(t) \Delta t \) that absorption occurs during \((t, t + \Delta t)\) is greater than or equal to the joint probability that the particle is near \( x_b \) at time \( t \) and the process increment \( \Delta X_t \equiv X(t + \Delta t) - X(t) \) carries the particle into the boundary. That is, to first-order in \( \Delta t \), the probability \( g(t) \Delta t \) must satisfy

\[
g(t) \Delta t \geq \mathbb{P}[X(t + \Delta t) - X(t) \equiv \Delta X_t > \Delta x, \quad x_b - \Delta x < X(t) < x_b]
\]

\[
= \mathbb{P}[X(t + \Delta t) - X(t) \equiv \Delta X_t > \Delta x \mid x_b - \Delta x < X(t) < x_b] \mathbb{P}[x_b - \Delta x < X(t) < x_b],
\]

where \( \Delta x \) is a small and arbitrary positive increment. Note that \( g(t) \) is the probability density function of the absorption time.

As \( \Delta t \to 0 \), the right-hand side of (14-82) approaches zero on the order of \( \sqrt{\Delta t} \) or slower if \( \Delta x \) is allowed to approach zero on the order of \( \sqrt{\Delta t} \). To see this, first note that (14-81)
implies

\[ P[ x_b - \Delta x < X(t) < x_b ] \geq \varepsilon \Delta x \tag{14-83} \]

so that

\[ g(t) \Delta t \geq \varepsilon \Delta x \quad P[ X(t + \Delta t) - X(t) \equiv \Delta X_t > \Delta x \mid x_b - \Delta x < X(t) < x_b ] . \tag{14-84} \]

However, a first-order-in-\( \Delta t \) approximation for the conditional variance of \( \Delta X_t \) is

\[ \text{Var}[ \Delta X_t \mid X(t) = x ] \approx \Delta t K^{(2)} , \tag{14-85} \]

where \( K^{(2)} \) is a positive constant (see the paragraph preceding (14-64)). With a non-zero probability, the magnitude of a random variable exceeds its standard deviation. Hence, (14-85) implies the existence of a \( p_0 > 0 \) such that

\[ P[ \Delta X_t > \sqrt{K^{(2)}} \Delta t \mid x_b - \Delta x < X(t) < x_b ] \geq p_0 > 0 \tag{14-86} \]

for sufficiently small \( \Delta t \). Set \( \Delta x = \sqrt{K^{(2)}} \Delta t \) in (14-84), and use (14-86) to obtain

\[ g(t) \Delta t \geq \varepsilon \sqrt{K^{(2)}} \Delta t p_0 , \tag{14-87} \]

which leads to

\[ g(t) \geq \frac{\varepsilon \sqrt{K^{(2)}}}{\sqrt{\Delta t}} p_0 . \tag{14-88} \]
Now, allow $\Delta t \to 0$ in (14-88) to obtain the contradiction that the density function $g(t)$ is infinite on $t_1 < t < t_2$. This contradiction implies that assumption (14-81) cannot be true; hence, boundary condition (14-80) must hold for all $t \geq 0$.

**Steady-State Solution to the Fokker-Planck Equation**

Many applications are of a time-invariant nature, and they are characterized by Fokker-Planck equations that have time-invariant intensity coefficients. Usually, such an application is associated with a Markov process $X(t)$ that becomes stationary as $t \to \infty$. As $t$ becomes large, the underlying density $f(x, t| x_0, t_0)$ that describes the process approaches a steady-state density function $f(x)$ that does not depend on $t$, $t_0$ or $x_0$. Often, the goal of system analysis in these cases is to find the steady-state density $f(x)$. Alternatively, in the steady state, the first and second moments of $X$ may be all that are necessary in some applications.

The steady-state density $f(x)$ satisfies

$$0 = - \frac{d}{dx} \left[ K^{(1)}(x) f(x) - \frac{1}{2} \frac{d}{dx} [K^{(2)} f(x)] \right],$$

the steady-state version of (14-64). In this equation, the diffusion coefficient $K^{(2)}$ is assumed to be constant (see the paragraph before (14-64)). Integrate both sides of (14-89) to obtain

$$\Im_{ss} = K^{(1)}(x) f(x) - \frac{1}{2} \frac{d}{dx} [K^{(2)} f(x)],$$

where constant $\Im_{ss}$ represents a steady-state value for the probability current.

The general solution of (14-90) can be found by using standard techniques. First, simplify the equation by substituting $y(x) \equiv K^{(2)} f(x)$ to obtain

$$\frac{dy}{dx} = 2 \left[ \frac{K^{(1)}(x)}{K^{(2)}} \right] y = -2 \Im_{ss}$$

(14-91)
The integrating factor for (14-91) is

\[
\mu(x) = \exp \left[ -2 \int^x \frac{K^{(1)}(\rho)}{K^{(2)}} \, d\rho \right].
\]  

(14-92)

This result and (14-91) can be used to write

\[
\mu \left[ \frac{dy}{dx} - 2 \left( \frac{K^{(1)}(x)}{K^{(2)}} \right) y \right] = \frac{d}{dx} [\mu y] = -2\mathfrak{I}_{ss} \mu
\]  

(14-93)

so that

\[
\frac{d}{dx} [\mu(x) f(x) K^{(2)}] = -2\mathfrak{I}_{ss} \mu(x).
\]  

(14-94)

Finally, the general solution to (14-90) can be written as

\[
f(x) = \left( \mu(x) K^{(2)} \right)^{-1} \left[ -2\mathfrak{I}_{ss} \int^x \mu(r) \, dr + C \right].
\]  

(14-95)

Note that (14-95) depends on the constants \(\mathfrak{I}_{ss}\) and \(C\). The steady-state value of probability current \(\mathfrak{I}_{ss}\) and the constant \(C\) are chosen so that \(f(x)\) satisfies specified boundary conditions (which are application-dependent) and the normalization condition

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1.
\]  

(14-96)

These results are used in what follows to compute the steady-state probability density function that describes the state variable in a first-order system driven by white Gaussian noise.
The One Dimensional First-Passage Time Problem

Suppose Markov process $X(t)$ denotes the instantaneous position of a particle that starts at $x_0$ when $t = 0$. Assume that absorbing boundaries exist at $b_1$ and $b_2$ with $b_1 < x_0 < b_2$. Let $t_a$ denote the amount of time that is required for the particle to reach an absorbing boundary for the first time. Time $t_a$ is called the first-passage time, and it varies from one path (sample function of $X(t)$) to the next. Hence, it is a random variable which depends on the initial value $x_0$. Figure 14-8 depicts the absorbing boundaries and two typical paths of the particle.

As it evolves with time, process $X(t)$ is described by the density $f(x,t \mid x_0,t_0)$, $t_0 = 0$. This evolution is described here in a qualitative manner; Fig. 14-9 is used in this effort, and it depicts the typical behavior of $f(x,t \mid x_0,0)$. At $t = 0$, the process starts at $x_0$; this implies that all of the probability is massed at $x_0$ as is shown by Fig. 14-9a. As $t$ increases, the location of the particle becomes more uncertain, and $f(x,t \mid x_0,0)$ tends to "spread out" in the $x$ variable as shown by Fig. 14-9b (this figure depicts the density for some $t = t_1 > 0$).

For $t > 0$, $f(x,t \mid x_0,0)$ is represented as the sum of a continuous function $q(x,t \mid x_0,0)$ and a pair of delta functions placed at $b_1$ and $b_2$. The delta functions represent the fact that probability will accumulate at the boundaries; this accumulation is due to the fact that sample functions will, sooner or later, terminate on a boundary and become absorbed. As shown by Fig. 14-9c which
depicts the case $t_2 > t_1$, $q$ continues to "spread out" as time increases, and it is more likely that the particle impacts a boundary and is absorbed. The area under $q(x, t \mid x_0, 0)$ decreases with time, and the delta function weights increase with time; however, the sum of the area under $q$ and the delta function weights is unity. As $t \to \infty$, the probability that the particle is absorbed approaches unity; this requirement implies that

$$\lim_{t \to \infty} q(x, t \mid x_0, 0) = 0. \quad (14-97)$$

This time-limiting case is depicted by Fig. 14-9d; on this figure, the quantity $q$ is zero, and the
delta function weights add to unity.

Function \( q(x,t \mid x_0,0) \) is a solution of the one-dimensional Fokker-Planck equation given by (14-58). The initial condition

\[
q(x,0 \mid x_0,0) = \delta(x - x_0)
\]  

(14-98)

and the absorbing boundary conditions

\[
q(b_1, t \mid x_0,0) = 0 \\
q(b_2, t \mid x_0,0) = 0,
\]  

(14-99)

t \geq 0, should be used in this effort.

**The Distribution and Density of the First-Passage Time Random Variable**

Function \( q(x,t \mid x_0,0) \) can be used to compute the distribution and density functions of the first-passage time random variable \( t_a \). First, the quantity

\[
\psi(t \mid x_0, 0) = \int_{b_1}^{b_2} q(x,t \mid x_0,0) dx ,
\]  

(14-100)

t \geq 0, represents the probability that the particle has not been absorbed by time \( t > 0 \) given that the position of the particle is \( x_0, b_1 < x_0 < b_2, \) at \( t = 0 \). In a similar manner,

\[
F_{t_a}(t \mid x_0, 0) = \mathbb{P}[ t_a \leq t \mid x_0, 0 ] \\
= 1 - \psi(t \mid x_0, 0) \\
= 1 - \int_{b_1}^{b_2} q(x,t \mid x_0,0) dx
\]  

(14-101)
represents the probability that the first-passage time random variable $t_a$ is not greater than $t$; that is, $F_{t_a}(t \mid x_0,0)$ is the distribution function for the first-passage time random variable $t_a$. Finally, the desired density function can be obtained by differentiating (14-101) with respect to $t$; this procedure yields the formula

$$f_{t_a}(t \mid x_0,0) = \frac{\partial}{\partial t} F_{t_a}(t \mid x_0,0)$$

(14-102)

for the density function of $t_a$.

**The Expected Value of the First-Passage Time Random Variable**

The expected value of the first-passage time random variable is important in many applications. A simple expression for $E[t_a]$ is determined in this section for the case where diffusion coefficient $K^{(2)}$ is constant.

The average value of the first-passage time can be expressed in terms of $q(x,t \mid x_0,0)$. To accomplish this, note that (14-102) can be used to write

$$E[t_a] = \int_0^\infty t f_{t_a}(t \mid x_0,0) dt = -\int_0^\infty t \left( \frac{\partial}{\partial t} \psi(t \mid x_0,0) \right) dt.$$  

(14-103)

However, the integral in (14-103) can be evaluated by parts to yield

$$E[t_a] = -t \psi(t \mid x_0,0) \bigg|_0^\infty + \int_0^\infty \psi(t \mid x_0,0) dt.$$  

(14-104)

Now, the integral in (14-104) is assumed to converge. Hence, it is necessary that $\psi(t \mid x_0,0)$ approach zero faster than $1/t$ as $t \to \infty$; this implies that the first term on the right of (14-104) is zero and that
\[ E[t_a] = \int_0^\infty \psi(t \mid x_0, 0) \, dt. \]  
(14-105)

Finally, substitute (14-100) into this and obtain

\[ E[t_a] = \int_{b_1}^{b_2} Q(x \mid x_0, 0) \, dx, \]  
(14-106)

where

\[ Q(x \mid x_0, 0) \equiv \int_0^\infty q(x, t \mid x_0, 0) \, dt. \]  
(14-107)

Note that (14-99) implies that \( Q \) satisfies the condition

\[ Q(b_1 \mid x_0, 0) = 0 \]  
(14-108)

\[ Q(b_2 \mid x_0, 0) = 0. \]

Equations (14-106) and (14-107) show that the expected value of the first-passage time can be expressed in terms of \( q(x, t \mid x_0, 0) \).

A simple first-order linear differential equation can be obtained that describes \( Q \). To obtain this equation, first note that \( q(x, t \mid x_0, 0) \) satisfies the one-dimensional Fokker-Planck equation

\[ \frac{\partial}{\partial t} q(x, t \mid x_0, 0) = -\frac{\partial}{\partial x} [K^{(1)}(x) q(x, t \mid x_0, 0)] + \frac{1}{2} K^{(2)} \frac{\partial^2}{\partial x^2} q(x, t \mid x_0, 0), \]  
(14-109)

where it has been assumed that \( K^{(2)} > 0 \) is constant. Now, integrate both sides of this last equation with respect to time and obtain
\[ q(x, \infty \mid x_0, 0) - q(x, 0 \mid x_0, 0) = -\frac{d}{dx}[K^{(1)}(x) Q(x \mid x_0, 0)] + \frac{1}{2} K^{(2)} \frac{d^2}{dx^2} Q(x \mid x_0, 0), \quad (14-110) \]

where \( Q \) is given by \( (14-107) \). Equations \( (14-97) \) and \( (14-98) \) can be used with \( (14-110) \) to obtain

\[ -\delta(x - x_0) = -\frac{d}{dx}[K^{(1)}(x) Q(x \mid x_0, 0)] + \frac{1}{2} K^{(2)} \frac{d^2}{dx^2} Q(x \mid x_0, 0). \quad (14-111) \]

Now, integrate both sides of this result to obtain

\[ \frac{d}{dx} Q(x \mid x_0, 0) - 2 \left( \frac{K^{(1)}(x)}{K^{(2)}} \right) Q(x \mid x_0, 0) = \frac{2}{K^{(2)}} \left( C_0 - U(x - x_0) \right), \quad (14-112) \]

where \( C_0 \) is a constant of integration, and \( U(x) \) denotes the unit step function. Equation \( (14-112) \) is a first-order, linear differential equation that describes \( Q \). Due to \( (14-108) \), it must be solved subject to the boundary conditions

\[ Q(b_1 \mid x_0, 0) \equiv \int_0^\infty q(b_1, t \mid x_0, 0) \, dt = 0 \quad (14-113) \]

\[ Q(b_2 \mid x_0, 0) \equiv \int_0^\infty q(b_2, t \mid x_0, 0) \, dt = 0. \]

The integrating factor \( \mu \) for \( (14-112) \) is related to the potential function \( U_p(x) \) (see \( (14-69) \)) by

\[ \mu(x) = \exp\left(-\frac{2}{K^{(2)}} \int^x K^{(1)}(\alpha) \, d\alpha\right) = \exp\left[U_p(x)\right] \quad (14-114) \]

since
\[
\frac{d}{dx} \left[ \mu(x) Q(x | x_0, 0) \right] = \mu(x) \left( \frac{2}{K^{(2)}} \right) \left( C_0 - U(x-x_0) \right) \tag{14-115}
\]

Now, integrate both sides of (14-115) to obtain

\[
Q(x | x_0, 0) = \mu(x)^{-1} \left[ \int_{b_1}^{b_2} \mu(\alpha) \left( C_0 - U(\alpha-x_0) \right) d\alpha + C_1 \right]. \tag{14-116}
\]

where \( C_1 \) is a constant of integration. Application of boundary conditions (14-113) leads to the determination that \( C_1 = 0 \) and

\[
C_0 = \frac{\int_{b_1}^{b_2} \mu(\alpha) U(\alpha-x_0) d\alpha}{\int_{b_1}^{b_2} \mu(\alpha) d\alpha}. \tag{14-117}
\]

A formula for the average first-passage time can be obtained by substituting (14-116), with \( C_1 = 0 \), into (14-106). This effort yields

\[
E[t_a] = \frac{2}{K^{(2)}} \int_{b_1}^{b_2} \mu(x)^{-1} \left[ \int_{b_1}^{x} \mu(\alpha) \left( C_0 - U(\alpha-x_0) \right) d\alpha \right] dx, \tag{14-118}
\]

where constant \( C_0 \) is given by (14-117).

**Boundary Absorption Rates**

Probability current flows into both boundaries \( b_1 \) and \( b_2 \). On Fig. 14-9, this is illustrated by the weights \( p_-(t) \) and \( p_+(t) \) increasing with time. However, in general, the flow is unequal, and one boundary may receive more probability current than the other. Hence, over a time interval \([0, T]\), the probability that flows into the boundaries may be unequal, a phenomenon that is analyzed in this section.

Figure 14-1 illustrates the fact that the amount of probability in an interval changes at a
rate that is equal to the current flowing into the interval. This implies that

\[ \int_{0}^{T} \Im (x_1, t \mid x_0, 0) \, dt \]

represents the total amount of probability that flows (in the direction of increasing x) past point \( x_1 \) during \([0, T]\). This result can be used to quantify the amount of probability that enters the boundaries depicted on Fig. 14-9.

As is illustrated on Fig. 14-9, probability flows into the boundaries at \( b_1 \) and \( b_2 \) as time increases. During the time interval \([0, T]\), the total amount of probability that flows into the boundaries \( b_1 \) and \( b_2 \) is

\[
\begin{align*}
p_-(T) &= -\int_{0}^{T} \Im (b_1, t \mid x_0, 0) \, dt \\
p_+(T) &= +\int_{0}^{T} \Im (b_2, t \mid x_0, 0) \, dt
\end{align*}
\]

(14-119)

respectively. The minus sign on the first of these integrals results from the fact that probability must flow in a direction of decreasing x (to the left) if it enters boundary \( b_1 \). As \( T \) approaches infinity, the total probability that enters the boundaries \( b_1 \) and \( b_2 \) is \( p_-(\infty) \) and \( p_+(\infty) = 1 - p_-(\infty) \), respectively.

In terms of \( q(x,t \mid x_0,0) \), the probability current on the interval \( b_1 \leq x \leq b_2 \) can be expressed as (see (14-65))

\[ \Im (x, t \mid x_0, 0) = K^{(1)}(x) \, q(x, t \mid x_0, 0) - \frac{1}{2} \frac{\partial}{\partial x} \{ K^{(2)}(x) \, q(x, t \mid x_0, 0) \} . \]  

(14-120)

Integrate this equation over \( 0 \leq t < \infty \), and use (14-107) to write
\[ \int_0^\infty \mathcal{Z}(x, t \mid x_0, 0) \, dt = K^{(1)}(x) \, Q(x \mid x_0, 0) - \frac{1}{2} \, \frac{d}{dx} [K^{(2)}(x \mid x_0, 0)]. \]  

(14-121)

This result describes the total probability that flows to the right (in the direction of increasing \( x \)) past point \( x \) during the time interval \([0, \infty)\). Now, in this last result, use boundary conditions \((14-108)\) and \( p_+ \) given by \((14-119)\) to write

\[ p_+(\infty) = -\frac{1}{2} \frac{d}{dx} [K^{(2)}(x \mid x_0, 0)] \bigg|_{x = b_2} \]  

(14-122)

for the total probability absorbed at boundary \( b_2 \). In a similar manner, the total probability absorbed at boundary \( b_1 \) is given as

\[ p_- (\infty) = \frac{1}{2} \frac{d}{dx} [K^{(2)}(x \mid x_0, 0)] \bigg|_{x = b_1} \]  

(14-123)

The sign difference in the last two equations results from the fact that current entering \( b_1 \) must flow in a direction that is opposite to the current flowing into boundary \( b_2 \).

An expression for the ratio of absorption probabilities \( p_+(\infty) \) and \( p_- (\infty) \) is needed in order to compute probabilities. By using \((14-122)\) and \((14-123)\), this ratio can be expressed as

\[ \frac{p_+(\infty)}{p_- (\infty)} = -\frac{\frac{d}{dx} [K^{(2)}(x \mid x_0, 0)] \bigg|_{x = b_2}}{\frac{d}{dx} [K^{(2)}(x \mid x_0, 0)] \bigg|_{x = b_1}} . \]  

(14-124)

The derivatives in this result can be supplied by \((14-112)\) when, as assumed here, \( K^{(2)} \) is independent of \( x \). Use boundary conditions \((14-113)\) in \((14-112)\) to obtain
\[
\frac{dq}{dx} Q(x \mid x_0, 0) \bigg|_{x = b_1, b_2} = \frac{2}{K(2)} \left( c_0 - U(x-x_0) \bigg|_{x = b_1, b_2} \right),
\]  

(14-125)

where \( c_0 \) is given by (14-117). Now, combine the last two equations to obtain

\[
\frac{p_+(\infty)}{p_-(\infty)} = -\frac{c_0 - U(b_2 - x_0)}{c_0 - U(b_1 - x_0)}
\]

(14-126)

which becomes

\[
\frac{p_+(\infty)}{p_-(\infty)} = -\frac{c_0 - 1}{c_0}
\]

(14-127)

for the usual \( b_1 < x_0 < b_2 \) case. This last result provides a ratio of the absorption probabilities in terms of the constant \( c_0 \). Finally, combine the equality \( p_+(\infty) + p_-(\infty) = 1 \) with (14-127) to obtain

\[
p_-(\infty) = c_0
\]

\[
p_+(\infty) = 1 - c_0
\]

(14-128)

for the total probabilities that enter the boundaries at \( b_1 \) and \( b_2 \), respectively.