Appendix 11B: Limit Superior, Limit Inferior and Limit of a Sequence of Sets (Events)

If \( A_1 \subset A_2 \subset A_3 \subset \ldots \) is a nested increasing sequence of events with

\[
A \equiv \bigcup_{n=1}^{\infty} A_n ,
\]

(11B1)

Theorem 11-1 states and proves the result

\[
\lim_{N \to \infty} P \left[ \bigcup_{n=1}^{N} A_n \right] = P \left[ \lim_{N \to \infty} \bigcup_{n=1}^{N} A_n \right] = P[A].
\]

(11B2)

That is, it is permissible to move the limit operation from “outside” to “inside” of the probability measure \( P \). A similar result was shown for a nested decreasing sequence of events. In this appendix, these simple continuity results are extended to sequences of arbitrary events.

Let \( A_1, A_2, \ldots \) be a sequence of arbitrary events (not necessarily nested in any manner). Define the events

\[
B_n \equiv \bigcup_{m=n}^{\infty} A_m , \quad \text{\( B_n \) is a nested decreasing sequence of events, and}
\]

(11B3)

\[
C_n \equiv \bigcap_{m=n}^{\infty} A_m , \quad \text{\( C_n \) is a nested increasing sequence of events.}
\]

Note that

\[
C_n \equiv \bigcap_{m=n}^{\infty} A_m \subseteq A_n \subseteq \bigcup_{m=n}^{\infty} A_m \equiv B_n
\]

(11B4)

for all \( n \).
As defined, $B_n$ and $C_n$ are legitimate events since countable intersections and unions of events are always events (recall the definition of a $\sigma$-algebra of events). Note that $B_n$ is a nested decreasing sequence of events, and $C_n$ is a nested increasing sequence of events. Because of this monotone nature, we can write

$$B_n = \bigcap_{m=1}^{n} B_m = \bigcap_{m=1}^{n} \bigcup_{m'=m}^{\infty} A_{m'}$$

(11B5)

$$C_n = \bigcup_{m=1}^{n} C_m = \bigcup_{m=1}^{n} \bigcap_{m'=m}^{\infty} A_{m'}$$

(11B6)

Now, define events $B$ and $C$ as the limits of $B_n$ and $C_n$, respectively; that is, define

$$B \equiv \lim_{n \to \infty} B_n = \bigcap_{m=1}^{\infty} B_m = \bigcap_{m=1}^{\infty} \bigcup_{m'=m}^{\infty} A_{m'}$$

(11B7)

$$C \equiv \lim_{n \to \infty} C_n = \bigcup_{m=1}^{\infty} C_m = \bigcup_{m=1}^{\infty} \bigcap_{m'=m}^{\infty} A_{m'}.$$

Note that

$$\rho \in B \implies \rho \text{ is in infinitely many of the } A_n$$

(11B8)

$$\rho \in C \implies \rho \text{ is in all but a finite number of the } A_n$$

(note: the phrase “infinitely many” is not the same as the phrase “all but a finite”).

That $B$ and $C$ exist as events follows from the fact that countable unions and intersections of events are events. That is, in the terminology of Chapter 1, $\sigma$-Algebra $\mathcal{F}$ is closed under countable intersections and unions (once you are “inside” $\mathcal{F}$ it’s hard to get “outside”).
In the literature, B is called the *limit supremum* (also called *lim sup*, *limit superior* or *upper limit*) of the sequence $A_n$, and it is denoted symbolically as

$$B = \limsup_{n \to \infty} A_n. \quad (11B9)$$

A little thought will lead to the conclusion that each element of B is in infinitely many of the $A_n$. That is, **if event B occurs then infinitely many of the $A_n$ occur** (we say that the $A_n$ occur *infinitely often*, or $A_n$ i.o.).

In the literature, C is called the *limit infimum* (also called *lim inf*, *limit inferior* or *lower limit*) of the sequence $A_n$, and it is denoted symbolically as

$$C = \liminf_{n \to \infty} A_n. \quad (11B10)$$

A little thought will lead to the conclusion that each element of C is in all but a finite number of the $A_n$. That is, **if event C occurs then all but a finite of the $A_n$ occur**.

Given a sequence of events $A_n$, then $B = \limsup A_n$ and $C = \liminf A_n$ always exist; however, they may not be equal. However, it is easily seen that $C \subset B$ always.

The infinite event sequence $A_1, A_2, \ldots$ is said to have a limit event A, and we write

$$A = \lim A_n, \quad (11B11)$$

if $\liminf A_n = \limsup A_n$. That is, the infinite event sequence $A_1, A_2, \ldots$ has limit event $A$ if events B and C, given by (11B6) and (11B7), respectively, are equal (*i.e.*, $B \subseteq C$ and $C \subseteq B$). In this case, we write

$$A = \lim A_n = \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n. \quad (11B12)$$
Given convergence of $A_n$ to $A$ as defined by (11B12), it is not difficult to show that

$$\lim_{n \to \infty} P(A_n) = P\left(\lim_{n \to \infty} A_n\right) = P(A)$$  \hspace{1cm} (11B13)

(this is a good homework problem!). So, in the sense described by (11B13), we can say that probability measures are continuous (often, (11B13) is taken as the definition of *sequential continuity*, and $P$ is said to be *sequentially continuous*).

Continuity property (11B13) held by $P$ is analogous to a well-known continuity property enjoyed by functions. Function $f(x)$ is continuous at $x = x_0$ if, and only if, $f(x_n) \to f(x_0)$ for any sequence (and all sequences) $x_n \to x_0$.

**Borel-Cantelli Lemma**

Let $A_1, A_2, \ldots$ be a sequence of events. Let

$$B = \limsup_{n \to \infty} A_n,$$

the event that infinitely many of the $A_n$ occur. Then it follows that

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(B) = 0.$$  \hspace{1cm} (11B14)

Furthermore, if all of the $A_i$ are independent (*i.e.*, we have an independent sequence) then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P(B) = 1.$$  \hspace{1cm} (11B15)

**Proof:** We show (11B14) first. Since $B = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \limsup_{n \to \infty} A_n$, we see that
B ⊂ \bigcup_{m=n}^{\infty} A_m, \quad (11B16)

for all n. This last equation implies that

\[ P(B) \leq \sum_{m=n}^{\infty} P(A_m) \quad (11B17) \]

for all n. Now, the hypothesis \( \sum_{n=1}^{\infty} P(A_n) < \infty \) implies that the right-hand side of (11B17) approaches zero as n approaches infinity. Hence, take the limit in (11B17) to see that \( P(B) = 0 \) as claimed by (11B14).

Now we show (11B15). From DeMorgans Law’s, we have

\[ \overline{B} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \overline{A}_m, \quad (11B18) \]

where over-bar denotes the complementation. However, as indexed on r, \( r \geq n \), \( \bigcap_{m=n}^{r} \overline{A}_m \) is a nested, decreasing sequence of sets. By Theorem 11-1 we have

\[ P\left( \bigcap_{m=n}^{\infty} \overline{A}_m \right) = \lim_{r \to \infty} P\left( \bigcap_{m=n}^{r} \overline{A}_m \right). \quad (11B19) \]

Due to the independence of the sequence, we have

\[ P\left( \bigcap_{m=n}^{\infty} \overline{A}_m \right) = \lim_{r \to \infty} P\left( \bigcap_{m=n}^{r} \overline{A}_m \right) = \lim_{r \to \infty} \prod_{m=n}^{r} P(\overline{A}_m) = \lim_{r \to \infty} \prod_{m=n}^{r} [1 - P(A_m)]. \quad (11B20) \]
For all $x \geq 0$, note that $1 - x \leq e^{-x}$. Apply this inequality to (11B20) and obtain

$$P\left(\bigcap_{m=n}^{\infty} \overline{A}_m\right) = \prod_{m=n}^{\infty} \exp\left[-P(A_m)\right] = \exp\left(-\sum_{m=n}^{\infty} P(A_m)\right)$$  \hspace{1cm} (11B21)

Use the hypothesis $\sum_{n=1}^{\infty} P(A_n) = \infty$ stated in (11B15) to see that (11B21) implies that

$$P\left(\bigcap_{m=n}^{\infty} \overline{A}_m\right) = 0$$  \hspace{1cm} (11B22)

for all $n$. Finally, from (11B18) we have

$$P(B) = \lim_{n \to \infty} P\left(\bigcap_{m=n}^{\infty} \overline{A}_m\right) = 0,$$  \hspace{1cm} (11B23)

so that $P(B) = 1$ as claimed by (11B15).♥

In general, (11B15) is false if the sequence of $A_i$ is not independent. To show this, consider any event $E$, $0 < P(E) < 1$, and define $A_n = E$ for all $n$. Then, $B = E$ and $P(B) = P(E) \neq 0$. 