Appendix 9-B: Random Poisson Points

As discussed in Chapter 1, let \( n(t_1,t_2) \) denote the number of Poisson random points in the interval \((t_1, t_2] \). The quantity \( n(t_1, t_2) \) is a non-negative-integer-valued random variable with

\[
P[n(t_1, t_2) = k] = p_k(\tau) = e^{-\lambda_d \tau} \frac{(\lambda_d \tau)^k}{k!}, \quad k \geq 0,
\]

where \( \tau = |t_1 - t_2| \) is the interval length, and constant \( \lambda_d > 0 \) is the average point density (average number of points per unit length). Note that Equation (9B1) does not depend on the absolute value of \( t_1 \) or \( t_2 \). We say that \( n(t_1, t_2) \) is Poisson distributed with parameter \( \lambda_d \tau \).

Random Poisson points have independent increments; the number of points in \((t_1, t_2]\) is independent of the number of points in \((t_3, t_4]\) if these two intervals do not overlap (i.e., the intersection of these intervals is the null set). This completes our first characterization of random Poisson points. In what follows, two other (equivalent) characterizations are discussed.

From the results of Chapter 2 of the class notes, in an interval of length \( \tau \), the expected number of points is \( \lambda_d \tau \). Somewhat surprisingly, this is also equal to the variance of \( n(t_1, t_2) \); we write

\[
E[n(t_1, t_2)] = \text{VAR}[n(t_1, t_2)] = \lambda_d \tau .
\]

A second characterization of random Poisson points has to do with the duration between points. As shown below, this duration is a random variable that is exponentially distributed. Random Poisson points can be characterized by their independent-increment nature as well as a requirement that the duration between points is described by an exponential random variable (this is our second characterization).

Finally, a third characterization can be given for random Poisson points. Random Poisson points can be characterized as an independent-increment point process that obeys the
Markov property (or memory-less property) that was discussed towards the end of Chapter 2 of these class notes (see Examples (2-14) and (2-15)). These alternative characterizations are discussed in this appendix.

Given the three characterizations mentioned above, it should be apparent that random Poisson points can be used to model many physical phenomena. They serve as a good model in many situations where something occurs repeatedly, the repetitions are independent of each other, and the average number of repetitions per unit time (or unit length) is constant over a large number of successive unit-time (or unit-length) intervals. Applications that involve N points placed randomly in an interval of length T can, when N and T are large, be modeled using Poisson points. As discussed in Chapter 1 of the class notes, a Poisson point model becomes exact as N and T approach infinity in such a manner that the ratio N/T approaches a constant \( \lambda_d \) (the average number of points per unit length). Poisson point models have been applied to problems dealing with electron emission in semiconductors and vacuum tubes (i.e., shot noise in electronic devices), telephone calls arriving at a switchboard and the arrival of cars at an intersection, to name a few applications.

**Exponentially Distributed Duration**

The duration between adjacent Poisson points is independent and exponentially distributed. First, we show a simpler result: we analyze the statistics of the duration from an arbitrary, but fixed, value of time to the nearest Poisson point. Let \( t_0 \) be any fixed marker/reference point in time; we show that the duration from \( t_0 \) to the nearest Poisson point (on either side of \( t_0 \)) is exponentially distributed.

Define random variable \( \tau_1 \) to be the duration from a fixed, but arbitrary, marker/reference point \( t_0 \) to the first Poisson point to the right of \( t_0 \) (see Fig. 9B-1). For algebraic variable \( \tau \), the event \([\tau_1 \leq \tau] \) is equivalent to the event “there are one or more Poisson points in the interval \((t_0, t_0 + \tau] \)”. That is, \([\tau_1 \leq \tau] = [n(t_0, t_0 + \tau) \geq 1] \) so that
We have

\[ F_{\tau_1}(\tau) = 1 - e^{-\lambda d \tau}, \quad \tau \geq 0, \]  

(9B4)

and,

\[ f_{\tau_1}(\tau) = \lambda d e^{-\lambda d \tau}, \quad \tau > 0, \]  

(9B5)

for the distribution and density functions, respectively, that describe random variable \( \tau_1 \). Equation (9B5) establishes the desired result that random variable \( \tau_1 \) is exponentially distributed. Now, let random variable \( \tau_{-1} \) denote the duration from \( t_0 \) back to the first Poisson point before \( t_0 \) (see Fig 9B-1). In a manner similar to that used above, one can show that \( \tau_{-1} \) is exponentially distributed.

The statistical properties of the duration from an arbitrary fixed point \( t_0 \) to any Poisson point (to the right or left of \( t_0 \)) can be analyzed. Define random variable \( \tau_n \) as the duration from \( t_0 \) to the \( n^{th} \) Poisson point to the right of \( t_0 \). The event \( [\tau_n \leq \tau] \) is equivalent to the event "there are \( n \) or more Poisson points in the interval \((t_0, t_0 + \tau]\)" so that
\[ P[\tau_n \leq \tau] = P[n(t_0, t_0 + \tau) \geq n] = 1 - P[n(t_0, t_0 + \tau) < n] = 1 - \sum_{k=0}^{n-1} \frac{\lambda_d^{(\tau)}^k}{k!} e^{-\lambda_d \tau}. \]  

(9B6)

We have

\[ F_{\tau_n}(\tau) = 1 - \sum_{k=0}^{n-1} \frac{\lambda_d^{(\tau)}^k}{k!} e^{-\lambda_d \tau}, \]

(9B7)

the distribution function for \( \tau_n \). Finally, differentiate (9B7) to obtain the gamma density function

\[ f_{\tau_n}(\tau) = \lambda_d \sum_{k=0}^{n-1} \frac{\lambda_d^{(\tau)}^k}{k!} - \sum_{k=1}^{n-1} \frac{\lambda_d^{(\tau)}^k}{(k-1)!} e^{-\lambda_d \tau} = \lambda_d \sum_{k=0}^{n-1} \frac{\lambda_d^{(\tau)}^k}{k!} - \sum_{k=0}^{n-2} \frac{\lambda_d^{(\tau)}^k}{k!} e^{-\lambda_d \tau} \]

(9B8)

\[ = \frac{\lambda_d^n}{(n-1)!} \tau^{n-1} e^{-\lambda_d \tau}, \quad \tau \geq 0. \]

(some authors refer to (9B8) as an Erlang density – see p. 459 of Stark and Woods, 4th edition).

This last result allows us to analyze the statistical properties of the duration between Poisson points. As shown by Figure 9B-1, define random variable \( T_n \equiv \tau_n - \tau_{n-1} \), the duration between the \( (n-1)^{th} \) and the \( n^{th} \) Poisson points to the right of arbitrary fixed point \( t_0 \). Since Poisson points in non-overlapping intervals are independent, the random variables \( T_n \) and \( \tau_{n-1} \) are independent, and the density function that describes \( \tau_n \) is the convolution of the densities that describe \( T_n \) and \( \tau_{n-1} \) (can you explain why this is true?). Equivalently, the moment generating function \( \Phi_{\tau_n}(s) \) for \( \tau_n \) is equal to the product \( \Phi_{T_n}(s) \Phi_{\tau_{n-1}}(s) \) of moment generating functions for \( T_n \) and \( \tau_{n-1} \), respectively (again, can you explain why this is true?). So, using (9B8) and the definition of the moment generating function, we calculate

\[ \Phi_{\tau_n}(s) = \int_{-\infty}^{\infty} f_{\tau_n}(x) e^{sx} \, dx = \frac{\lambda_d^n}{(n-1)!} \int_{0}^{\infty} x^{n-1} e^{-\lambda_d x} e^{sx} \, dx = \frac{\lambda_d^n}{(n-1)!} \int_{0}^{\infty} x^{n-1} e^{-(\lambda_d-s)x} \, dx. \]  

(9B9)
The last integral on the right of (9B9) is tabulated as

\[
\int_0^\infty x^{n-1} e^{-(\lambda_d-s)x} \, dx = \frac{\Gamma(n)}{(\lambda_d-s)^n} = \frac{(n-1)!}{(\lambda_d-s)^n},
\]

where \( \Gamma(n) = (n-1)! \) is the well known Gamma function with integer \( n \) argument. Finally, substitute (9B10) into (9B9) to obtain

\[
\Phi_{\tau_n}(s) = \frac{\lambda_d^n}{(n-1)!} \frac{\Gamma(n)}{(\lambda_d-s)^n} = \frac{\lambda_d^n}{(\lambda_d-s)^n}. \tag{9B11}
\]

Finally, since \( \Phi_{\tau_n}(s) = \Phi_{T_n}(s)\Phi_{\tau_{n-1}}(s) \), we have

\[
\Phi_{T_n}(s) = \frac{\Phi_{\tau_n}(s)}{\Phi_{\tau_{n-1}}(s)} = \frac{\lambda_d^n}{(\lambda_d-s)^n} \left[ \frac{\lambda_d^{n-1}}{(\lambda_d-s)^{n-1}} \right]^{-1} = \frac{\lambda_d}{(\lambda_d-s)}. \tag{9B12}
\]

Note that \( \Phi_{T_n}(s) \) is the moment generating function of an exponential random variable. Hence, random variable \( T_n \), the duration between the \((n-1)\)th and \( n \)th Poisson points to the right of \( t_0 \) (or any two Poisson points), is described by an exponential random variable with parameter \( \lambda_d \).

**Poisson Points Obey the Markov Property**

Poisson points obey the Markov property (see the System Reliability section in Chapter 2). As above, denote \( t_0 \) as a fixed (but arbitrary) marker/reference point. Denote \( \tau_1 \) as a random variable that describes the time duration between \( t_0 \) and the next Poisson point to the right, random variable \( \tau_1 \) being described by density \( f_{\tau_1}(\tau) \). Let conditional density \( f(\tau' | \tau_1 > \tau) \) describes random variable \( \tau_1 \) conditioned on the event \([\tau_1 > \tau] \), \( \tau \) an algebraic variable. Since \( f_{\tau_1}(\tau) \) is exponential, we have the Markov property
as shown in Chapter 2 (see Examples (2-14) and (2-15)). A similar formula can be written for the (conditional) density that describes random variable \( T_n \) conditioned on the event \( T_n > \tau \). The fact that it has been very long (or short) \( \tau \) seconds since \( t_0 \) and not observing a point does not make the next Poisson point more (or less) likely. Stated differently, the probability of finding a point in the interval \((\tau, \tau')\] is not influenced by the length of \((t_0, \tau]\), an interval containing no point(s).

**Development of (9B13)**

Recall that \( \tau_1 \) denotes the distance from any marker/reference \( t_0 \) to the first Poisson point after \( t_0 \). It is exponentially distributed with parameter \( \lambda > 0 \). The distribution of \( \tau_1 \) conditioned on the event \( \tau_1 > \tau \) is

\[
F(\tau' | \tau_1 > \tau) = \frac{P[\tau_1 \leq \tau', \tau_1 > \tau]}{P[\tau_1 > \tau]} = \frac{P[\tau < \tau_1 \leq \tau']}{1 - P[\tau_1 \leq \tau]} = \frac{f_{\tau_1}(\tau') - f_{\tau_1}(\tau)}{1 - F_{\tau_1}(\tau)}, \quad \tau' \geq \tau \geq 0
\]

\[
= 0, \quad \text{otherwise}
\]

Distribution functions \( F_{\tau_1}(\tau) \) and \( F(\tau' | \tau_1 > \tau) \) are right continuous. As a result, as \( \tau' \to \tau^+ \), \( F(\tau' | \tau_1 > \tau) \to F(\tau | \tau_1 > \tau) = 0 \). The density of \( \tau_1 \) conditioned on the event \( \tau_1 > \tau \) is

\[
f(\tau' | \tau_1 > \tau) = \frac{d}{d\tau'} F(\tau' | \tau_1 > \tau) = \frac{f_{\tau_1}(\tau')}{1 - F_{\tau_1}(\tau)}, \quad \tau' > \tau \geq 0
\]

\[
= 0, \quad \text{otherwise}
\]

Note that \( f(\tau' | \tau_1 > \tau) \to \lambda \) as \( \tau' \to \tau^+ \).

Recall that \( \tau_1 \) is exponentially distributed; the distribution (9B4) and density (9B5) are repeated here as
and,

\[ f_{\tau_1}(\tau) = \lambda_d e^{-\lambda_d \tau}, \quad \tau > 0, \quad (9B17) \]

respectively. Substitute (9B17) into (9B15) to obtain

\[
f(\tau' | \tau_1 > \tau) = \frac{\lambda_d e^{-\lambda_d \tau'} e^{-\lambda_d \tau}}{e^{-\lambda_d \tau}} \\
= \lambda_d e^{-\lambda_d (\tau'-\tau)} \\
= f_{\tau_1}(\tau' - \tau) \quad \tau' > \tau \geq 0.
\]

(9B18)

Starting from any arbitrary marker/reference point \( t_0 \), the fact that it has been very long (or short) \( \tau \) seconds since \( t_0 \) and not observing a point does not make the next Poisson point more (or less) likely. Stated differently, the fact that no point is in \( (t_0, \tau] \), \( \tau > t_0 \), does not influence what will be found in \( (\tau, \tau'], \tau > \tau \). Stated again, the fact that no point is in \( (t_0, \tau] \), \( \tau > t_0 \), is forgotten when it comes to characterizing the points in \( (\tau, \tau'], \tau > \tau \), an attribute known as the memory-less, or Markov, property.

We have just shown that exponential random variable \( \tau_1 \) has the Markov property (as do all exponential random variables). The converse is true as well. If random variable \( \tau_1 \) has the Markov property, it must be exponentially distributed. To show this, start by assuming that \( \tau_1 \) has the Markov property (9B13), and read again the sentence following (9B15) to conclude

\[
\lim_{\tau' \to \tau^+} \frac{f_{\tau_1}(\tau')}{1-F_{\tau_1}(\tau)} = \lambda
\]

(9B19)
for some constant $\lambda > 0$. However, this last result implies

$$\lim_{\tau' \to \tau^+} \frac{dF_{\tau_1}(\tau')}{d\tau} = -\lambda F_{\tau_1}(\tau) + \lambda.$$  \hfill (9B20)

Since $F_{\tau_1}(0) = 0$, this last differential equation implies that $F_{\tau_1}(\tau) = 1 - e^{-\lambda \tau}$, $\tau \geq 0$, so that $\tau_1$ is exponentially distributed. Independent increments, an average of $\lambda_d$ points per unit length, and the Markov property are three attributes that imply/characterize Poisson points.