Appendix 4.2: Hermitian Matrices

A square $n \times n$ matrix $B$ is said to be Hermitian if $B^* = B$. Here, the $*$ denotes complex-conjugate transpose (some authors use an “H” as a subscript to denote complex-conjugate transpose, and they would write $B^H = B$). We need two important attributes of Hermitian matrices. First, their eigenvalues are always real-valued. Secondly, they are unitary similar to a diagonal matrix containing the eigenvalues. That is, there exists an $n \times n$ unitary matrix $U$ (i.e., $U^* = U^{-1}$ or $U^*U = I$) such that $U^*BU$ is a diagonal matrix with the eigenvalues of $B$ on its diagonal.

**Theorem 4.2.1**

A Hermitian matrix has real-valued eigenvalues.

**Proof:** Let $\lambda$ and $\vec{X}$ be an eigenvalue and eigenvector, respectively, of $B$. On the left, multiply $B\vec{X} = \lambda\vec{X}$ by $\vec{X}^*$ to obtain

$$\vec{X}^*B\vec{X} = \lambda\vec{X}^*\vec{X} \quad (4.2-1)$$

Now, $\vec{X}^*\vec{X}$ is always real-valued. Also, $\vec{X}^*B\vec{X}$ has to be real-valued since $(\vec{X}^*B\vec{X})^* = \vec{X}^*B^*\vec{X} = \vec{X}^*B\vec{X}$. By inspection of the last equation, we conclude that $\lambda$ must be real-valued.♥

Let $B$ be a Hermitian matrix ($B^* = B$). As we know, it has real eigenvalues. Also, it is unitary similar to a diagonal matrix containing eigenvalues on the diagonal.

**Theorem 4.2.2**

Let $n \times n$ matrix $B$ be Hermitian. Then there exists an $n \times n$ unitary matrix $U$ (i.e., $U^*U = I$) such that $U^{-1}BU = U^*BU$ is a diagonal matrix with the eigenvalues of $B$ on its diagonal. Furthermore, the eigenvectors of $B$ are the columns of $U$.

**Proof:** By the previous theorem, there exists an $n \times n$ unitary matrix $U$ such that $U^*BU$ is upper-triangular with the eigenvalues on the diagonal (i.e., $U^*BU$ is in Schur Form). However, $(U^*BU)^* = U^*B^*U = U^*BU$, so $U^*BU$ is Hermitian. Note that an upper-triangular, Hermitian
matrix must necessarily be diagonal. Hence \( D \equiv U^*BU \) is diagonal with eigenvalues on its diagonal. From \( BU = DU \), it is clear that the columns of \( U \) are the eigenvectors of \( B \).

According to Theorem 4.2-1, matrix \( D \) has only real-valued eigenvalues. According to Theorem 4.2-2, matrix \( D \) looks like

\[
D = U^*BU = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_r \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

(4.2-2)

where \( \lambda_k \), \( 1 \leq k \leq r \leq n \), are the non-zero eigenvalues. If \( \lambda_k > 0 \), \( 1 \leq k \leq n \), then Hermitian \( B \) is said to be positive definite, and this is denoted by writing \( B > 0 \). In this case, it is easily shown by using the diagonal decomposition that \( \bar{X}^*B\bar{X} > 0 \) for all \( \bar{X} \neq \vec{0} \). If \( \lambda_k \geq 0 \), \( 1 \leq k \leq n \), then Hermitian \( B \) is said to be nonnegative definite, and this is denoted by writing \( B \geq 0 \). In this case, it is easily shown by using the diagonal decomposition that \( \bar{X}^*B\bar{X} \geq 0 \) for all \( \bar{X} \). Clearly, the positive definite attribute is "stronger" than the nonnegative definite attribute. A positive definite matrix is also nonnegative definite, but the converse is not true.

**MatLab Example: Schur Decomposition of a Hermitian Matrix**

```matlab
% Enter a Hermitian matrix B
>B = [1 2+3j 5j; 2-3j 2 1+7j; -5j 1-7j 3]
B = [1  (2+3j)  +5j
     (2-3j)  2   (1+7j)
     -5j   (1-7j)  3 ]

% The eigenvalues of B are real-valued (as is true for the eigenvalues of any Hermitian matrix!).
>eig(B)
```


12.4327, -0.5018, -5.9309

> [U,T]=schur(B)

\[
U = \begin{bmatrix}
0.4514 & (-0.4251 - 0.6376j) & (-0.4489 - 0.0864j) \\
(0.5023 - 0.3105j) & (0.2035 + 0.4896j) & (-0.2789 - 0.5407j) \\
(-0.1772 - 0.6450j) & (-0.1264 - 0.3401j) & (0.5034 - 0.4093j)
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
12.4327 & 0 & 0 \\
0 & -0.5018 & 0 \\
0 & 0 & -5.9309
\end{bmatrix}
\]

% It works! The Schur Form For Hermitian B is a diagonal matrix with the eigenvalues on the diagonal. ♥

**Analogy Between Hermitian Matrices and Real Numbers**

An analogy between Hermitian matrices and real numbers can be made. Each positive (alternatively, nonnegative) real number has a positive (alternatively, nonnegative) square root. A similar statement can be made for Hermitian matrices.

**Theorem 4.2.3**

An \(n \times n\) Hermitian matrix \(H\) is positive (alternatively, nonnegative) definite if, and only if, there exists a positive (alternatively, nonnegative) definite Hermitian matrix \(H_0\) such that \(H_0^2 = H\). Matrix \(H_0\) is called the *square root* of \(H\).

**Proof:** (We prove the positive definite case; the nonnegative definite case is similar.)

Suppose that \(H\) is Hermitian and positive definite. Then it has eigenvalues \(\lambda_i, 1 \leq i \leq n\), that are real and positive. Furthermore, there exists a unitary matrix \(U\) such that \(U^*HU = D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\), a diagonal matrix with the eigenvalues of \(H\) on its diagonal. The square root of \(D\) is

\[
D_0 = \sqrt{D} = \text{diag}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\}, \tag{4.2-3}
\]
a diagonal matrix with eigenvalues $\lambda_k$, $1 \leq k \leq n$, on its diagonal. Note that matrix $H$ can be written as

$$H = UD_0D_0^* = (UD_0U^*)(UD_0U^*) = H_0^2,$$  \hspace{1cm} (4.2-4)

where $H_0 = UD_0U^*$. We write $H_0 = \sqrt{H}$, and call $H_0$ the *square root* of Hermitian $H$. Note that the eigenvalues of $H_0$ are $\sqrt{\lambda_k}$, $1 \leq k \leq n$, all positive. Hence $H_0$ is a positive definite Hermitian matrix.

Conversely, suppose that $H = H_0^2$, where $H_0$ is a positive definite Hermitian matrix. Clearly, $H$ is Hermitian; we show that $H$ is positive definite. Let $\tilde{X}$ and $\lambda$ be an eigenvector and eigenvalue pair for $H_0$; note that $\lambda$ is positive since $H_0$ is positive definite. Then we have

$$H\tilde{X} = H_0^2\tilde{X} = H_0(\lambda\tilde{X}) = \lambda^2\tilde{X},$$  \hspace{1cm} (4.2-5)

so that $\tilde{X}$ and $\lambda^2$ is an eigenvector and eigenvalue pair for $H$. Hermitian matrix $H$ has positive eigenvalues. Hence, it is positive definite. ♥